

MULTICOMMUTATORS AND MULTIPLIER THEOREMS

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ABSTRACT. For all $k, n \in \mathbb{N}$, we obtain the boundedness of the n th dimensional Calderón-Coifman-Journé-type multicommutator $\mathcal{C}_k^{(n)}(f, a_1, \dots, a_k)(x)$, given by

$$\text{p.v.} \int_{\mathbb{R}^{nk}} f(x - y_1 - \dots - y_k) \left(\prod_{j=1}^k \frac{1}{(y_{j1}y_{j2} \dots y_{jn})^2} \int_{x_1-y_{j1}}^{x_1} \dots \int_{x_n-y_{jn}}^{x_n} a_j(\vec{u}_j) d\vec{u}_j \right) dy_1 \dots dy_k.$$

from $L^{p_0}(\mathbb{R}^n) \times L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_k}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ in the largest possible open set of indices $(1/p_0, 1/p_1, \dots, 1/p_k)$ with $1/p_0 + 1/p_1 + \dots + 1/p_k = 1/p$, which is the range $1/(k+1) < p < \infty$. Our proof exploits the limited smoothness of the symbol of the multicommutator via a new multilinear multiplier theorem for symbols of restricted smoothness which lie locally in certain Sobolev spaces. Our multiplier approach to this problem is a new contribution in the understanding of Calderón's commutator program.

1. COMMUTATORS AND MULTICOMMUTATORS

In 1965 Calderón [2] introduced the (first-order) commutator

$$(1.1) \quad \mathcal{C}_1(f; a)(x) = \text{p.v.} \int_{\mathbb{R}} \frac{A(x) - A(y)}{(x - y)^2} f(y) dy = \pi \left[\frac{d}{dx} H, A \right](f)(x),$$

where a is the derivative of a Lipschitz function A and f is a test function on the real line. Here $[T, A](f)$ denotes the commutator $T(Af) - AT(f)$ of an operator T with the multiplication operator $f \mapsto Af$ and H is the classical Hilbert transform on the real line. The higher-order analogues of (1.1) for $k \geq 1$ are given by

$$(1.2) \quad \mathcal{C}_k(f; a)(x) = \text{p.v.} \int_{\mathbb{R}} \left(\frac{A(x) - A(y)}{x - y} \right)^k \frac{f(y)}{x - y} dy,$$

and are linear in the input function f but not linear in the function A . The higher-order commutators in (1.2) have a very important connection with the Cauchy integral along a Lipschitz curve $\{(y, A(y)) : y \in \mathbb{R}\}$. To obtain the L^2 boundedness of the Cauchy integral, Coifman, McIntosh, and Meyer [8] derived L^2 estimates for (1.2) with bound

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at most $C(1+k)^4 \|A'\|_{L^\infty(\mathbb{R})}^k$. These were deduced from $L^2(\mathbb{R}) \times L^\infty(\mathbb{R}) \times \cdots \times L^\infty(\mathbb{R})$ bounds for the following multilinearization of (1.2)

$$(1.3) \quad \mathcal{C}_k^{(1)} = \text{p.v.} \int_{\mathbb{R}} \frac{A_1(x) - A_1(y)}{x - y} \cdots \frac{A_k(x) - A_k(y)}{x - y} \frac{f(y)}{x - y} dy,$$

where $a_j = A'_j$, $j = 1, \dots, k$, by restricting on the diagonal $a_1 = \cdots = a_k$.

Note that (1.3) is equal to a constant multiple of the iterated commutator

$$\left[\cdots \left[\left[\frac{d^k}{dx^k} H, A_1 \right], A_2 \right], \cdots, A_k \right].$$

The operator $\mathcal{C}_k^{(1)}$ is too singular to fall under the scope of multilinear Calderón-Zygmund theory [21]. However it was shown to be bounded from $L^{p_0}(\mathbb{R}) \times \cdots \times L^{p_k}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1 < p_0, \dots, p_k < \infty$ and $(1/p_0 + \cdots + 1/p_k)^{-1} = p > 1/(k+1)$; see C. Calderón [3] when $k = 1$, Coifman and Meyer [9] when $k = 2$, Duong, Grafakos, and Yan [12] when $k \geq 3$. In the case $p > 1$, Muscalu [31] studied this operator via time-frequency analysis. The sharpest bound concerning the Calderón-Coifman-Meyer multicommutator $\mathcal{C}_k^{(1)}$ is the endpoint estimate

$$\mathcal{C}_k^{(1)} : \underbrace{L^1(\mathbb{R}) \times \cdots \times L^1(\mathbb{R})}_{k+1 \text{ times}} \rightarrow L^{1/(k+1), \infty}(\mathbb{R}),$$

which was proved in [3] when $k = 1$, in [9] when $k = 2$, and in [12] when $k \geq 3$.

Higher dimensional versions of the Calderón-Coifman-Meyer multicommutators defined by

$$(1.4) \quad \begin{aligned} & \mathcal{C}_k^{(n)}(f, a_1, \dots, a_k)(x) \\ &= \text{p.v.} \int_{\mathbb{R}^n} f(y) \left(\prod_{l=1}^n \frac{1}{(y_l - x_l)^{k+1}} \right) \prod_{j=1}^k \int_{x_1}^{y_1} \cdots \int_{x_n}^{y_n} a_j(u_1, \dots, u_n) du_1 \cdots du_n dy, \end{aligned}$$

where f is a function on \mathbb{R}^n , and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. The operator $\mathcal{C}_k^{(n)}$ was introduced by a suggestion of Coifman when $n = 2$ and $k = 1$. The $L^2 \times L^\infty \rightarrow L^2$ bound for $\mathcal{C}_1^{(2)}$ was first obtained by Aguirre [1]. In his celebrated work on product-type spaces, Journé [25] named $\mathcal{C}_1^{(2)}$ a *bicommutator of Calderón-Coifman type*, and in [26] he showed that

$$(1.5) \quad \|\mathcal{C}_k^{(2)}(f, \underbrace{a, \dots, a}_{k\text{-times}})\|_{L^2(\mathbb{R}^2)} \leq C_\delta (1+k)^{2+\delta} \|a\|_{L^\infty(\mathbb{R}^2)}^k \|f\|_{L^2(\mathbb{R}^2)}$$

for any $\delta > 0$. In view of Journé's contribution in the study of $\mathcal{C}_k^{(2)}$, we call the operator $\mathcal{C}_k^{(n)}$ in (1.4) the *higher-dimensional Calderón-Coifman-Journé multicommutator*. They

were studied by Aguirre [1], Journé [25], and also by Christ and Journé [6]. The boundedness of $\mathcal{C}_k^{(1)}$ on L^p for $p \geq 1$ was also studied by Muscalu [32] for all k via time-frequency analysis. Off-diagonal estimates for the latter were recently obtained by Seeger, Smart, and Street [36].

It turns out that the multiplier of $\mathcal{C}_k^{(n)}$ has only limited smoothness which cannot be handled by classical multiplier theorems. To understand this type of limited smoothness, we study a variant of $\mathcal{C}_k^{(n)}$ with similar product-type singularities. Precisely, we study the *Calderón-Coifman-Journé type multicommutator* $\mathcal{C}_k^{(n)}(f, a_1, \dots, a_m)(x)$ given by

$$(1.6) \quad \text{p.v.} \int_{\mathbb{R}^{nk}} f(x - y_1 - \dots - y_k) \left(\prod_{j=1}^k \frac{1}{(y_{j1}y_{j2} \dots y_{jn})^2} \int_{x_1 - y_{j1}}^{x_1} \dots \int_{x_n - y_{jn}}^{x_n} a_j(\vec{u}_j) d\vec{u}_j \right) dy_1 \dots dy_k,$$

Note that like $\mathcal{C}_k^{(n)}$, $\mathcal{C}_k^{(n)}$ also coincides with the classical Calderón commutator when $k = 1$, but for $k \geq 2$, $\mathcal{C}_k^{(n)}$ and $\mathcal{C}_k^{(n)}$ are different.

We denote by $\widehat{g}(\xi) = \int_{\mathbb{R}^n} g(x) e^{-2\pi i x \cdot \xi} dx$ the Fourier transform of a function g on \mathbb{R}^n and by $g^\vee(x) = \widehat{g}(-x)$ its inverse Fourier transform. Viewed as a $(k+1)$ -linear Fourier multiplier operator acting on the tuple (f, a_1, \dots, a_k) , $\mathcal{C}_k^{(1)}(f, a_1, \dots, a_k)(x)$ can also be written as

$$(-i\pi)^k \int_{\mathbb{R}^{k+1}} \left(\widehat{f}(\xi_0) \prod_{j=1}^k \widehat{a}_j(\xi_j) \right) \left[\prod_{j=1}^k \text{sgn}(\xi_j) \Phi(\xi_0/\xi_j) \right] e^{2\pi i x(\xi_0 + \dots + \xi_k)} d\xi_0 \dots d\xi_k,$$

where Φ is the following Lipschitz function on the real line:

$$(1.7) \quad \Phi(s) = \begin{cases} -1, & \text{if } s \leq -1; \\ 2s + 1, & \text{if } -1 < s < 0; \\ 1, & \text{otherwise.} \end{cases}$$

The Calderón-Coifman-Journé type multicommutator in (1.6) is quite singular. This can be seen by expressing it in $(k+1)$ -linear Fourier multiplier form as follows:

$$(1.8) \quad \mathcal{C}_k^{(n)}(f, a_1, \dots, a_k)(x) = \int_{\mathbb{R}^{(k+1)n}} e^{2\pi i x \cdot (\xi_0 + \xi_1 + \dots + \xi_k)} \sigma_k^{(n)}(\xi_0, \xi_1, \dots, \xi_k) \widehat{f}(\xi_0) \prod_{j=1}^k \widehat{a}_j(\xi_j) d\xi_0 d\xi_1 \dots d\xi_k,$$

where $\xi_j = (\xi_{j1}, \dots, \xi_{jn})$, $j = 0, \dots, k$ with symbol (or multiplier) being

$$(1.9) \quad \sigma_k^{(n)}(\xi_0, \xi_1, \dots, \xi_k) = (-i\pi)^{kn} \prod_{j=1}^k \prod_{l=1}^n \left(\text{sgn}(\xi_{jl}) \Phi\left(\frac{\xi_{0l}}{\xi_{jl}}\right) \right),$$

and Φ is as in (1.7). It follows from this formula that the singularities of the multiplier are spread along an increasing number of affine subspaces as k increases. Boundedness

for $\mathcal{C}_1^{(n)}$ from $L^{p_1} \times L^{p_2}$ to L^p for $p > 1/2$, can be derived by Muscalu's work on Calderón commutators on polydiscs [33, Theorem 6.1] via time-frequency analysis.

As observed, the combinatorial complexity of the multiplier increases with k . Nevertheless, we are able to prove the boundedness of $\mathcal{C}_k^{(n)}$ on $L^p(\mathbb{R}^n)$ in all dimensions for all $k \geq 1$ in the largest open set of indices possible, i.e., $1/(k+1) < p < \infty$. Our main theorem is as follows:

Theorem 1.1. *For all $k, n \geq 1$ there is a constant $C(n, k)$ such that if $1 < p_j < \infty$, $j = 0, \dots, k$ and $1/p = 1/p_0 + 1/p_1 + \dots + 1/p_k$ and $1/(k+1) < p < \infty$, we have*

$$\|\mathcal{C}_k^{(n)}(f_0, f_1, \dots, f_k)\|_{L^p(\mathbb{R}^n)} \leq C(n, k) \prod_{j=0}^k \|f_j\|_{L^{p_j}(\mathbb{R}^n)}$$

for all Schwartz functions f_j .

We prove this theorem using a new multiplier theorem for multilinear operators with product-type local Sobolev conditions and a delicate and careful analysis of $\sigma_k^{(n)}$ in order to determine the best L^r -based Sobolev space to which it belongs locally, exploiting to the maximum the limited smoothness of the function Φ . We obtain boundedness in the largest possible open set of indices via an analysis that relies solely on the study of the corresponding multiplier; this approach is novel in this context.

A few words about notation. The norm of a multilinear operator T from the product of spaces $X_1 \times \dots \times X_m$ to X is denoted by $\|T\|_{X_1 \times \dots \times X_m \rightarrow X}$. We will use arrows, such as $\vec{\xi}$, to indicate elements of \mathbb{R}^{mn} . We also use the notation $A \lesssim B$ to indicate that A is bounded by a constant times B , where the constant is independent of any essential parameters.

2. MULTILINEAR MULTIPLIER THEORY

The theory of multilinear multipliers has made significant advances in recent years. An n -dimensional m -linear multiplier is a bounded function σ on $(\mathbb{R}^n)^m$ associated with an m -linear operator T_σ on $\mathbb{R}^n \times \dots \times \mathbb{R}^n$ in the following way:

$$T_\sigma(f_1, \dots, f_m)(x) = \int_{(\mathbb{R}^n)^m} \widehat{f}_1(\xi_1) \cdots \widehat{f}_m(\xi_m) \sigma(\xi_1, \dots, \xi_m) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_m)} d\xi_1 \cdots d\xi_m,$$

where f_j , $j = 1, \dots, m$, are Schwartz functions in \mathbb{R}^n , and $\widehat{f_j}(\xi_j) = \int_{\mathbb{R}^n} f_j(x) e^{-2\pi i x \cdot \xi_j} dx$ is the Fourier transform of f_j . A classical result of Coifman and Meyer [10, 11] says that if for all sufficiently large multiindices $\alpha_1, \dots, \alpha_m \in (\mathbb{Z}^+ \cup \{0\})^n$ we have

$$(2.1) \quad \left| \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_m}^{\alpha_m} \sigma(\xi_1, \dots, \xi_m) \right| \lesssim (|\xi_1| + \dots + |\xi_m|)^{-(|\alpha_1| + \dots + |\alpha_m|)}$$

for all $(\xi_1, \dots, \xi_m) \in (\mathbb{R}^n)^m \setminus \{(0, \dots, 0)\}$, then T_σ admits a bounded extension from $L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ when $1 < p_1, \dots, p_m \leq \infty$, $1/p = 1/p_1 + \dots + 1/p_m$, and $1 \leq p < \infty$. The extension of this theorem to indices $p > 1/m$ was simultaneously obtained by Kenig and Stein [28] (when $m = 2$) and Grafakos and Torres [21]. This theorem provides an m -linear extension of Mikhlin's classical linear multiplier result [29]. Hörmander [24] obtained an improvement of Mikhlin's theorem showing that when $m = 1$, T_σ maps $L^{p_1}(\mathbb{R}^n)$ to $L^{p_1}(\mathbb{R}^n)$, $1 < p_1 < \infty$ under the weaker condition

$$(2.2) \quad \sup_{j \in \mathbb{Z}} \|(I - \Delta)^{s/2} (\sigma(2^j \cdot) \widehat{\Psi})\|_{L^2(\mathbb{R}^n)} < \infty,$$

where $s > n/2$ and $\widehat{\Psi}$ is a smooth function supported in an annulus centered at the origin. Here Δ is the Laplacian and $(I - \Delta)^{s/2}$ is an operator given on the Fourier transform side by multiplication with $(1 + 4\pi^2|\xi|^2)^{s/2}$. Hörmander's theorem was extended to L^r -based Sobolev spaces and to indices $p_1 \leq 1$, with L^{p_1} replaced by the Hardy space H^{p_1} , by Calderón and Torchinsky [4].

The adaptation of Hörmander's theorem to the multilinear setting was first obtained by Tomita [39]. This theorem was later extended by Grafakos and Si [20] to the range $p < 1$ by replacing L^2 -based Sobolev spaces by L^r -based Sobolev spaces. The endpoint cases where some p_j are equal to infinity were treated by Grafakos, Miyachi, and Tomita [18]. Fujita and Tomita [13] provided weighted extensions of these results and also noticed that the operator $(I - \Delta)^{s/2}$ in $(\mathbb{R}^n)^m$ can be replaced by $(I - \Delta_1)^{s_1/2} \dots (I - \Delta_m)^{s_m/2}$, where Δ_j is the Laplacian in the ξ_j th variable. The bilinear version of the Calderón and Torchinsky theorem was proved by Miyachi and Tomita [30], while the m -linear version (for general m) was proved by Grafakos and Nguyen [16] and Grafakos, Miyachi, Nguyen, and Tomita [17].

The Calderón-Coifman-Journé-type multicommutators are associated with multipliers which are of limited smoothness in every variable, precisely, they do not have enough smoothness to be differentiated more than once in each variable. To cover this situation we need a theorem that handles symbols on $(\mathbb{R}^n)^m$ which, for instance, have one derivative in each variable but no more than two derivatives in a given variable. We notice that in the case where s_j are positive integers for all j , replacing $(I - \Delta)^{s/2}$ in $(\mathbb{R}^n)^m$ by $(I - \Delta_1)^{s_1/2} \dots (I - \Delta_m)^{s_m/2}$ as in Fujita and Tomita [13], essentially reflects the following decay condition for the derivatives of σ

$$(2.3) \quad |\partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \dots \partial_{\xi_m}^{\beta_m} \sigma(\xi_1, \dots, \xi_m)| \lesssim (|\xi_1| + \dots + |\xi_m|)^{-\sum_{j=1}^m |\beta_j|},$$

where each multiindex β_j is restricted to $|\beta_j| \leq s_j$. For our purposes, we need a coordinate-wise version of (2.3), i.e., we need to further relax (2.3) as follows:

$$(2.4) \quad \left| \partial_{\xi_{11}}^{\beta_{11}} \dots \partial_{\xi_{1n}}^{\beta_{1n}} \partial_{\xi_{21}}^{\beta_{21}} \dots \partial_{\xi_{2n}}^{\beta_{2n}} \dots \partial_{\xi_{m1}}^{\beta_{m1}} \dots \partial_{\xi_{mn}}^{\beta_{mn}} \sigma(\xi_1, \dots, \xi_m) \right| \\ \lesssim (|\xi_1| + \dots + |\xi_m|)^{-\sum_{j=1}^m \sum_{\ell=1}^n \beta_{j\ell}},$$

where $0 \leq \beta_{j\ell} \leq s_j$ and $\xi_j = (\xi_{j1}, \dots, \xi_{jn})$. Condition (2.4) weakens the Coifman-Meyer hypothesis (2.1) in the sense that it does not allow any one-dimensional variable to be differentiated more than a certain number of times. Since in our applications $s_j > 0$ are fractional, naturally, we need a Sobolev-space version of (2.4).

Another extension of the Coifman-Meyer multiplier theorem is in the multiparameter setting. In this case, the expression on the right in (2.1) is replaced by the product of the sums of the absolute values of each coordinate, i.e., (2.1) is relaxed to

$$(2.5) \quad |\partial_{\vec{\xi}}^{\vec{\alpha}} \sigma(\vec{\xi})| \lesssim (|\xi_{11}| + \dots + |\xi_{m1}|)^{-(\alpha_{11} + \dots + \alpha_{m1})} \dots (|\xi_{1n}| + \dots + |\xi_{mn}|)^{-(\alpha_{1n} + \dots + \alpha_{mn})}$$

for sufficiently large multiindices $\alpha_1, \dots, \alpha_m$ with $\vec{\alpha} = (\alpha_1, \dots, \alpha_m)$ and $\vec{\xi} = (\xi_1, \dots, \xi_m)$. Such a condition was first considered by Muscalu, Pipher, Tao, and Thiele [34, 35], who obtained boundedness for the associated operator in the largest open set of exponents possible, i.e., from $L^{p_1} \times \dots \times L^{p_m}$ to L^p with $1/p_1 + \dots + 1/p_m = 1/p$ and $1/m < p < \infty$.

To accommodate the study of the multicommutator, we need a multilinear multiplier theorem that extends both (2.4) and (2.5) and is also valid for fractional derivatives, i.e., it holds for multipliers which lie locally in Sobolev spaces. In its classical derivative formulation, the condition we need is:

$$(2.6) \quad \left| \partial_{\xi_{11}}^{\beta_{11}} \dots \partial_{\xi_{1n}}^{\beta_{1n}} \partial_{\xi_{21}}^{\beta_{21}} \dots \partial_{\xi_{2n}}^{\beta_{2n}} \dots \partial_{\xi_{m1}}^{\beta_{m1}} \dots \partial_{\xi_{mn}}^{\beta_{mn}} \sigma(\xi_1, \dots, \xi_m) \right| \\ \lesssim (|\xi_{11}| + \dots + |\xi_{m1}|)^{-(\beta_{11} + \dots + \beta_{m1})} \dots (|\xi_{1n}| + \dots + |\xi_{mn}|)^{-(\beta_{1n} + \dots + \beta_{mn})},$$

where $\beta_{j1}, \dots, \beta_{jn} \leq s_j$ for all $j = 1, \dots, m$ and $\xi_j = (\xi_{j1}, \dots, \xi_{jn}) \in \mathbb{R}^n$. Naturally (2.6) extends both (2.4) and (2.5).

We introduce some notation. Let $d \in \mathbb{Z}^+$ and $\vec{\gamma} = (\gamma_1, \dots, \gamma_d) \in \mathbb{R}^d$. Given $t \geq 0$, we say that $\vec{\gamma} > t$ if $\gamma_i > t$ for all $i = 1, \dots, d$. For $\vec{\gamma} = (\gamma_1, \dots, \gamma_d) > 0$ we define a weight $\omega_{\vec{\gamma}}$ on \mathbb{R}^d ($d \in \mathbb{Z}^+$) as follows:

$$\omega_{\vec{\gamma}}(t_1, \dots, t_d) = \prod_{i=1}^d (1 + |t_i|^2)^{\frac{\gamma_i}{2}}.$$

We denote by $L_{\vec{\gamma}}^r(\mathbb{R}^d)$ the Sobolev space of all functions F on \mathbb{R}^d such that

$$\|F\|_{L_{\vec{\gamma}}^r} := \left\| (\omega_{\vec{\gamma}} \widehat{F})^\vee \right\|_{L^r(\mathbb{R}^d)} < \infty.$$

We now formulate our multilinear multiplier result for bounded functions σ having m variables in \mathbb{R}^{nl} . These are the symbols of m -linear operators acting on m -tuples of functions defined on \mathbb{R}^{nl} .

Theorem 2.1. *Let $m, n, l \in \mathbb{Z}^+$ and $\vec{\gamma} = (\bar{\gamma}_{11}, \dots, \bar{\gamma}_{1l}, \bar{\gamma}_{21}, \dots, \bar{\gamma}_{2l}, \dots, \bar{\gamma}_{m1}, \dots, \bar{\gamma}_{ml}) \in \mathbb{R}^{nml}$, where $\bar{\gamma}_{ij} = (\gamma_{ij1}, \dots, \gamma_{ijn}) \in (\mathbb{R}^+)^n$ for each $1 \leq i \leq m$ and $1 \leq j \leq l$. Suppose that $\max_{1 \leq j \leq l} \max_{1 \leq k \leq n} (1, \frac{1}{\gamma_{ijk}}) < r \leq 2$ for all $i = 1, \dots, m$ and let σ be a bounded function of the variables $((\xi_{11}, \dots, \xi_{1l}), (\xi_{21}, \dots, \xi_{2l}), \dots, (\xi_{m1}, \dots, \xi_{ml})) \in (\mathbb{R}^{nl})^m$ such that*

$$(2.7) \quad \sup_{k_1, \dots, k_l \in \mathbb{Z}} \left\| \sigma(\Xi_{k_1, \dots, k_l}) \prod_{j=1}^l \widehat{\Psi}(\xi_{1j}, \xi_{2j}, \dots, \xi_{mj}) \right\|_{L_{\vec{\gamma}}^r(\mathbb{R}^{nml})} = A < \infty,$$

where $\xi_{ij} \in \mathbb{R}^n$ for $1 \leq i \leq m$, $1 \leq j \leq l$,

$$\Xi_{k_1, \dots, k_l} = (2^{k_1} \xi_{11}, \dots, 2^{k_l} \xi_{1l}, 2^{k_1} \xi_{21}, \dots, 2^{k_l} \xi_{2l}, \dots, 2^{k_1} \xi_{m1}, \dots, 2^{k_l} \xi_{ml}),$$

and $\widehat{\Psi}$ is a smooth function supported in the annulus $\frac{1}{2} \leq |(\xi_{11}, \xi_{21}, \dots, \xi_{m1})| \leq 2$ in $(\mathbb{R}^n)^m$ satisfying

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}(\xi_{11}, \xi_{21}, \dots, \xi_{m1})) = 1, \quad \text{for all } (\xi_{11}, \xi_{21}, \dots, \xi_{m1}) \neq 0.$$

If $\max_{1 \leq j \leq l} \max_{1 \leq k \leq n} (1, \frac{1}{\gamma_{ijk}}) < p_i < \infty$ for all $i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, then the m -linear multiplier operator T_σ associated with the symbol σ satisfies

$$(2.8) \quad \|T_\sigma\|_{L^{p_1}(\mathbb{R}^{nl}) \times \dots \times L^{p_m}(\mathbb{R}^{nl}) \rightarrow L^p(\mathbb{R}^{nl})} \lesssim A.$$

A version of this result was given Chen and Lu [5] when $r = 2$ and when the Sobolev space $L_{\vec{\gamma}}^2(\mathbb{R}^{nml})$ is replaced by a subspace of it, in which a combined smoothness of order $\sum_{i,j} \gamma_{ijk}$ is required in each of the l variables in \mathbb{R}^{mn} ; in this case all these derivatives could fall on a single real coordinate $\xi_{i_0 j_0 k}$ of the multiplier. In contrast to this, recall that in the Sobolev space $L_{\vec{\gamma}}^r(\mathbb{R}^{nml})$ at most γ_{ijk} derivatives fall on each real coordinate ξ_{ijk} . This is crucial in the study of the multicommutator $\mathcal{C}_k^{(n)}$ whose symbol $\sigma_k^{(n)}$ has smoothness limited to $1 - \varepsilon$ derivatives on every real coordinate for any $\varepsilon > 0$. Also for this problem, it is necessary to take r close to 1 in order to obtain the full range of exponents, i.e., $p_i > 1$ for all i .

The following corollary provides a weakening of condition (2.4) and follows by taking $l = 1$ in the preceding theorem:

Corollary 2.2. *Let $\vec{\gamma} = (\bar{\gamma}_1, \dots, \bar{\gamma}_m) > 0$, where $\bar{\gamma}_i = (\gamma_{i1}, \dots, \gamma_{in})$ for each $1 \leq i \leq m$. Suppose that $\max(1, \frac{1}{\gamma_{i\ell}}) < r \leq 2$ for all $i = 1, \dots, m$ and $\ell = 1, \dots, n$ and let σ be a*

bounded function on \mathbb{R}^{mn} such that

$$(2.9) \quad \sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\Psi} \right\|_{L_{\vec{\gamma}}^r(\mathbb{R}^{mn})} = A < \infty,$$

where $\widehat{\Psi}$ is a smooth function supported in the annulus $\frac{1}{2} \leq |(\xi_1, \dots, \xi_m)| \leq 2$ in \mathbb{R}^{mn} satisfying

$$\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}(\xi_1, \dots, \xi_m)) = 1, \quad \text{for all } (\xi_1, \dots, \xi_m) \neq 0.$$

If $\max_{1 \leq \ell \leq n} (1, \frac{1}{\gamma_{i\ell}}) < p_i < \infty$ for all $i = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$, then we have

$$(2.10) \quad \|T_\sigma\|_{L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim A.$$

The next corollary follows by taking $n = 1$ in Theorem 2.1 (and then replacing l by n). It corresponds to condition (2.5) for classical derivatives.

Corollary 2.3. *Let $1 < r \leq 2$ and $\vec{\gamma} = (\bar{\gamma}_1, \dots, \bar{\gamma}_m) > 1/r$, where $\bar{\gamma}_j = (\gamma_{j1}, \dots, \gamma_{jn})$ for each $1 \leq j \leq m$. Suppose that σ is a bounded function on \mathbb{R}^{mn} such that $A < \infty$, where*

$$\sup_{k_1, \dots, k_n \in \mathbb{Z}} \left\| \sigma(2^{k_1} \xi_{11}, \dots, 2^{k_n} \xi_{1n}, \dots, 2^{k_1} \xi_{m1}, \dots, 2^{k_n} \xi_{mn}) \prod_{\ell=1}^n \widehat{\Psi}(\xi_{1\ell}, \dots, \xi_{m\ell}) \right\|_{L_{\vec{\gamma}}^r(\mathbb{R}^{mn})} = A,$$

and $\widehat{\Psi}$ is a smooth function on \mathbb{R}^m supported in the annulus $\frac{1}{2} \leq |\eta| \leq 2$ satisfying

$$(2.11) \quad \sum_{k \in \mathbb{Z}} \widehat{\Psi}(2^{-k} \eta) = 1, \quad \text{for all } \eta \in \mathbb{R}^m \setminus \{0\}.$$

Then for $\max_{1 \leq \ell \leq n} (\frac{1}{\gamma_{j\ell}}, 1) < p_j < \infty$ for all $j = 1, \dots, m$, and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ we have

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim A.$$

An immediate corollary of Corollary 2.3 is the following:

Corollary 2.4. *Let $\sigma_\ell(\xi_{1\ell}, \dots, \xi_{m\ell})$ be bounded functions on \mathbb{R}^m for $1 \leq \ell \leq n$. Let $\sigma(\xi_1, \dots, \xi_m) = \prod_{\ell=1}^n \sigma_\ell(\xi_{1\ell}, \dots, \xi_{m\ell})$, where $\xi_j = (\xi_{j1}, \dots, \xi_{jn}) \in \mathbb{R}^n$, $1 \leq j \leq m$. Suppose that for some $\vec{\gamma}$ and r as in Corollary 2.3 we have*

$$(2.12) \quad \sup_{1 \leq \ell \leq n} \sup_{k \in \mathbb{Z}} \left\| \sigma_\ell(2^k \cdot) \widehat{\Psi} \right\|_{L_{\gamma_{1\ell}, \dots, \gamma_{m\ell}}^r(\mathbb{R}^m)} = A < \infty$$

where $\widehat{\Psi}$ is a smooth function supported in an annulus in \mathbb{R}^m that satisfies (2.11). Then for $\max_{1 \leq \ell \leq n} (\frac{1}{\gamma_{j\ell}}, 1) < p_j < \infty$ for all $j = 1, \dots, m$ and $\frac{1}{p} = \frac{1}{p_1} + \dots + \frac{1}{p_m}$ we have

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim A^n.$$

We also have a version of Corollary 2.4 in which fewer than m variables appear in each function σ_ℓ .

Corollary 2.5. *For each $1 \leq \ell \leq n$ let $m_\ell \leq m$ and let σ_ℓ be bounded functions on \mathbb{R}^{m_ℓ} . If $\sigma(\xi_1, \dots, \xi_m) = \prod_{\ell=1}^n \sigma_\ell(\xi_{S_\ell, \ell})$ with $\xi_{S_\ell, \ell} = (\xi_{a_1 \ell}, \xi_{a_2 \ell}, \dots, \xi_{a_{m_\ell} \ell})$ and $S_\ell = \{a_1, a_2, \dots, a_{m_\ell}\}$ is a subset of $\{1, 2, \dots, m\}$ of size m_ℓ . Suppose that for some $\vec{\gamma}$ and r as in Corollary 2.3 we have*

$$A = \sup_{1 \leq \ell \leq n} \sup_{j \in \mathbb{Z}} \|\sigma_\ell(2^j \cdot) \Psi\|_{L_{\vec{\gamma}}^r(\mathbb{R}^{m_\ell})} < \infty.$$

Then we have

$$\|T_\sigma\|_{L^{p_1}(\mathbb{R}^n) \times \dots \times L^{p_m}(\mathbb{R}^n) \rightarrow L^p(\mathbb{R}^n)} \lesssim A^n.$$

3. PRELIMINARIES FOR THE PROOF OF THEOREM 2.1

We begin by discussing some lemmas concerning the weight $\omega_{\vec{\gamma}}$.

Lemma 3.1. *Let $1 < r < \infty$ and $\vec{\gamma} > 0$ be a vector in \mathbb{R}^d . Assume that σ is a function defined on \mathbb{R}^d , supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq M\}$ and set $K = \sigma^\vee$, the inverse Fourier transform of σ . Then there exists a constant $C_{d,M}$ such that*

$$(3.1) \quad \|K\omega_{\vec{\gamma}}\|_{L^\infty(\mathbb{R}^d)} \leq C_{d,M} \|K\omega_{\vec{\gamma}}\|_{L^r(\mathbb{R}^d)}.$$

Proof. Let φ be a Schwartz function on \mathbb{R}^d such that $\widehat{\varphi}(\xi) = 1$ for all $\xi \in \mathbb{R}^d$, $|\xi| \leq M$. Then we have $\sigma(\xi) = \sigma(\xi)\widehat{\varphi}(\xi)$. Taking the inverse Fourier transform yields

$$K(x) = (K * \varphi)(x) = \int_{\mathbb{R}^d} K(x-y)\varphi(y) dy.$$

Since $\omega_{\vec{\gamma}}(x) \lesssim \omega_{\vec{\gamma}}(x-y)\omega_{\vec{\gamma}}(y)$ for all $y \in \mathbb{R}^d$, we have for every $x \in \mathbb{R}^d$

$$\begin{aligned} |K(x)|\omega_{\vec{\gamma}}(x) &= \omega_{\vec{\gamma}}(x) \left| \int_{\mathbb{R}^d} K(x-y)\varphi(y) dy \right| \\ &\lesssim \int_{\mathbb{R}^d} \omega_{\vec{\gamma}}(x-y) |K(x-y)| \omega_{\vec{\gamma}}(y) |\varphi(y)| dy \\ &\lesssim \|\omega_{\vec{\gamma}}(x-\cdot)K(x-\cdot)\|_{L^r(\mathbb{R}^d, dy)} \|\omega_{\vec{\gamma}}\varphi\|_{L^{r'}(\mathbb{R}^d, dy)} \\ &\lesssim \|K\omega_{\vec{\gamma}}\|_{L^r(\mathbb{R}^d, dy)}, \end{aligned}$$

where we used Hölder's inequality in the penultimate inequality and the implicit constant in the last inequality depends only on φ which relies only on the support of σ , i.e., the constant M in the statement of the lemma and the dimension d . This proves (3.1). \square

Interpolating between (3.1) and the trivial estimate

$$\|K\omega_{\vec{\gamma}}\|_{L^r(\mathbb{R}^d)} \leq \|K\omega_{\vec{\gamma}}\|_{L^r(\mathbb{R}^d)}$$

yields

$$(3.2) \quad \|K\omega_{\vec{\gamma}}\|_{L^q(\mathbb{R}^d)} \lesssim \|K\omega_{\vec{\gamma}}\|_{L^r(\mathbb{R}^d)},$$

for all $1 < r \leq q \leq \infty$.

Now assume that $1 \leq \rho < r \leq 2$ which of course implies $r' \leq \rho' \leq \infty$. Using (3.2) first and then applying Young's inequality yields

$$(3.3) \quad \|K\omega_{\vec{\gamma}}\|_{L^{\rho'}(\mathbb{R}^d)} \lesssim \|K\omega_{\vec{\gamma}}\|_{L^{r'}(\mathbb{R}^d)} \lesssim \|\sigma\|_{L^r_{\vec{\gamma}}(\mathbb{R}^d)},$$

where K and σ are as in Lemma 3.1 related via $\widehat{K} = \sigma$.

For $\vec{\gamma} = (\gamma_1, \dots, \gamma_d) > 0$ define a differential operator on \mathbb{R}^d by setting

$$J^{\vec{\gamma}}(f)(\xi_1, \dots, \xi_d) = (I - \partial_{\xi_1}^2)^{\frac{\gamma_1}{2}} \cdots (I - \partial_{\xi_d}^2)^{\frac{\gamma_d}{2}} f(\xi_1, \dots, \xi_d).$$

Lemma 3.2. (i) Let φ be a Schwartz function on \mathbb{R}^d , let $\vec{\gamma} = (\gamma_1, \dots, \gamma_d) > 0$, and let $1 < r < \infty$. Then

$$(3.4) \quad \|\sigma\varphi\|_{L^r_{\vec{\gamma}}(\mathbb{R}^d)} \lesssim \|\sigma\|_{L^r_{\vec{\gamma}}(\mathbb{R}^d)} = \|J^{\vec{\gamma}}(\sigma)\|_{L^r(\mathbb{R}^d)}$$

for all $\sigma \in L^r_{\vec{\gamma}}(\mathbb{R}^d)$.

(ii) If $f \in L^r_{\vec{\gamma}}(\mathbb{R}^d)$ and $t > 0$ is given, then $f(t \cdot) \in L^r_{\vec{\gamma}}(\mathbb{R}^d)$.

Proof. We prove (i) first. Recall that $\omega_{\vec{\gamma}}(\xi) = \prod_{j=1}^d (1 + |\xi_j|^2)^{\frac{\gamma_j}{2}}$ for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$. For fixed $x \in \mathbb{R}^d$ we have

$$\begin{aligned} |J^{\vec{\gamma}}(\sigma\varphi)(x)| &= \left| \int_{\mathbb{R}^d} \omega_{\vec{\gamma}}(\xi) (\widehat{\sigma} * \widehat{\varphi})(\xi) e^{2\pi i x \cdot \xi} d\xi \right| \\ &= \left| \int_{\mathbb{R}^d} \omega_{\vec{\gamma}}(\xi) \left\{ \int_{\mathbb{R}^n} \widehat{\sigma}(\xi - \eta) \widehat{\varphi}(\eta) d\eta \right\} e^{2\pi i x \cdot \xi} d\xi \right| \\ &= \left| \int_{\mathbb{R}^d} \widehat{\varphi}(\eta) \left\{ \int_{\mathbb{R}^d} \omega_{\vec{\gamma}}(\xi + \eta) \widehat{\sigma}(\xi) e^{2\pi i x \cdot \xi} d\xi \right\} e^{2\pi i x \cdot \eta} d\eta \right| \\ (3.5) \quad &\leq \int_{\mathbb{R}^d} |\widehat{\varphi}(\eta)| \left| \int_{\mathbb{R}^d} F_{\vec{\gamma}}^{\eta}(\xi) \widehat{J^{\vec{\gamma}}(\sigma)}(\xi) e^{2\pi i x \cdot \xi} d\xi \right| d\eta, \end{aligned}$$

where $F_{\vec{\gamma}}^{\eta}(\xi) = \frac{\omega_{\vec{\gamma}}(\xi + \eta)}{\omega_{\vec{\gamma}}(\xi)}$. We now claim that $F_{\vec{\gamma}}^{\eta}$ is an L^r multiplier on \mathbb{R}^d for any fixed $\vec{\gamma} > 0$ and $\eta \in \mathbb{R}^d$ with multiplier norm at most a multiple of $(1 + |\eta|^2)^{c(\vec{\gamma})}$, where $c(\vec{\gamma}) > 0$. More precisely, we claim that $F_{\vec{\gamma}}^{\eta}$ satisfies the estimate

$$(3.6) \quad \left(\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} F_{\vec{\gamma}}^{\eta}(\xi) \widehat{J^{\vec{\gamma}}(\sigma)}(\xi) e^{2\pi i x \cdot \xi} d\xi \right|^r dx \right)^{\frac{1}{r}} \leq C(1 + |\eta|^2)^{c(\vec{\gamma})} \|\sigma\|_{L^r_{\vec{\gamma}}(\mathbb{R}^d)}$$

for all $1 < r < \infty$. Then applying Minkowski's inequality in (3.5) and combining with (3.6) yields (3.4).

To verify the validity of (3.6), we note that $\frac{1}{1+|a|^2} \leq \frac{1+|t+a|^2}{1+|t|^2} \leq 1 + |a|^2$ and that, by taking derivatives directly, there exist two positive constants $C_\delta = 1 + 2\delta$ and $c(\delta) = |\delta - 1| + 1$ such that

$$(3.7) \quad \left| \frac{d^k}{dt^k} \left(\frac{1 + |t + a|^2}{1 + t^2} \right)^\delta \right| \leq C_\delta \frac{(1 + |a|^2)^{c(\delta)}}{|t|^k}$$

for $k \in \{0, 1\}$, $\delta > 0$, $a \in \mathbb{R}$ and $0 \neq t \in \mathbb{R}$. This guarantees that $F_{\vec{\gamma}}^\eta$ is a Marcinkiewicz multiplier with constant $C(1 + |\eta|^2)^{c(\vec{\gamma})}$, where $c(\vec{\gamma}) = \sum_{j=1}^d c(\gamma_j)$. Applying Theorem 6.2.4 in [14], (3.6) follows.

Part (ii) is a consequence of the fact that $\prod_{j=1}^d \left(\frac{1+t^2|\xi_j|^2}{1+|\xi_j|^2} \right)^{\gamma_j/2}$ is a Mihlin-Hömander multiplier, with norm depending on $t > 0$. \square

Let us indicate variables in \mathbb{R}^{nl} by (z_1, \dots, z_n) , where each z_j lies in \mathbb{R}^l . Fix a smooth function $\widehat{\Psi}$ supported in an annulus in \mathbb{R}^l satisfying $\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j}z) = 1$ for all $z \neq 0$. For $j \in \mathbb{Z}$, define a Littlewood-Paley operator

$$\Delta_j^{(k)}(f) = (\widehat{f}(z_1, z_2, \dots, z_n) \widehat{\Psi}(2^{-j}z_k))^\vee$$

acting on functions f on \mathbb{R}^{nl} . We need the following result.

Lemma 3.3. *For $f \in L^p(\mathbb{R}^{nl})$ with $1 < p < \infty$ we have*

$$(3.8) \quad \left\| \left(\sum_{j_1} \cdots \sum_{j_n} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)}(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{nl})} \lesssim \|f\|_{L^p(\mathbb{R}^{nl})}.$$

Conversely, for $0 < p < \infty$ there exists a constant C such that for any f in $L^2(\mathbb{R}^{nl})$ satisfying $\|(\sum_{j_1} \cdots \sum_{j_n} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)}(f)|^2)^{1/2}\|_{L^p} < \infty$ we have

$$(3.9) \quad \|f\|_{L^p(\mathbb{R}^{nl})} \leq C \left\| \left(\sum_{j_1} \cdots \sum_{j_n} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)}(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^{nl})}.$$

Proof of Lemma 3.3. The proof of (3.8) is well known and is omitted; see for instance [14, Theorem 6.1.6] when $l = 1$ but the same idea works for all l . So we now focus on (3.9) which we prove inductively. The case $n = 1$ is the reverse of the Littlewood-Paley inequality when $p > 1$. When $n = 1$ and $p \leq 1$, then by [15, Theorem 2.2.9] there is a polynomial Q on \mathbb{R}^l such that

$$\|f - Q\|_{H^p(\mathbb{R}^l)} \lesssim \left\| \left(\sum_{j_1} |\Delta_{j_1}^{(1)}(f)|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^l)} < \infty.$$

Since f lies in $L^2(\mathbb{R}^l)$, it follows that $f - Q$ is a locally integrable function which lies in $H^p(\mathbb{R}^l)$ and thus $\|f - Q\|_{L^p} \lesssim \|f - Q\|_{H^p(\mathbb{R}^l)} < \infty$. Therefore $Q = 0$ and (3.9) follows.

Assume that the assertion is valid for n . We will prove the case $n + 1$. Let r_k be the Rademacher functions reindexed by $k \in \mathbb{Z}$. Applying (3.9) to $g = \sum_k f_k r_k$ we obtain

$$\begin{aligned} & \int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} \left(\sum_k |f_k(x_1, \dots, x_n)|^2 \right)^{p/2} dx_1 \cdots dx_n \\ & \lesssim \int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} \int_0^1 \left| \sum_k f_k(x_1, \dots, x_n) r_k(t_{n+1}) \right|^p dt_{n+1} dx_1 \cdots dx_n \\ & = C \int_0^1 \int_{\mathbb{R}^l} \cdots \int_{\mathbb{R}^l} |g(x_1, \dots, x_n)|^p dx_1 \cdots dx_n dt_{n+1} \end{aligned}$$

where we used the property of Rademacher functions; see for instance [14, Appendix C]. By the induction hypothesis, the preceding expression is bounded by a multiple of

$$\begin{aligned} & \int_0^1 \int_{(\mathbb{R}^l)^n} \left(\sum_{j_1} \cdots \sum_{j_n} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} g(x_1, \dots, x_n)|^2 \right)^{p/2} dx_1 \cdots dx_n dt_{n+1} \\ & \lesssim \int_0^1 \int_{(\mathbb{R}^l)^n} \int_{[0,1]^n} \left| \sum_{j_1} \cdots \sum_{j_n} \Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} g(x_1, \dots, x_n) \prod_{i=1}^n r_{j_i}(t_i) \right|^p dt_1 \cdots dt_n d\vec{x} dt_{n+1} \\ & \approx \int_{(\mathbb{R}^l)^n} \int_{[0,1]^{n+1}} \left| \sum_{j_1, \dots, j_n, k} \Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} f_k(x_1, \dots, x_n) r_k(t_{n+1}) \prod_{i=1}^n r_{j_i}(t_i) \right|^p dt_1 \cdots dt_{n+1} d\vec{x} \\ & \lesssim \int_{(\mathbb{R}^l)^n} \left(\sum_{j_1} \cdots \sum_{j_n} \sum_k |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} f_k(x_1, \dots, x_n)|^2 \right)^{p/2} dx_1 \cdots dx_n, \end{aligned}$$

where we used properties of Rademacher functions and $d\vec{x} = dx_1 \cdots dx_n$. It follows that

$$\begin{aligned} & \int_{(\mathbb{R}^l)^{n+1}} |f(x_1, \dots, x_{n+1})|^p dx_1 \cdots dx_{n+1} \\ & \lesssim \int_{(\mathbb{R}^l)^{n+1}} \sup_{t>0} |[\varphi_t * f(x_1, \dots, x_n)](x_{n+1})|^p dx_{n+1} dx_1 \cdots dx_n \\ & \lesssim \int_{(\mathbb{R}^l)^{n+1}} \left(\sum_{j_{n+1}} |\Delta_{j_{n+1}}^{(n+1)} f(x_1, \dots, x_{n+1})|^2 \right)^{p/2} dx_{n+1} dx_1 \cdots dx_n \\ & \approx \int_{(\mathbb{R}^l)^{n+1}} \left(\sum_{j_{n+1}} |\Delta_{j_{n+1}}^{(n+1)} f(x_1, \dots, x_{n+1})|^2 \right)^{p/2} dx_1 \cdots dx_{n+1} \\ & \lesssim \int_{(\mathbb{R}^l)^{n+1}} \left(\sum_{j_1} \cdots \sum_{j_n} \sum_{j_{n+1}} |\Delta_{j_1}^{(1)} \cdots \Delta_{j_n}^{(n)} \Delta_{j_{n+1}}^{(n+1)} f(x_1, \dots, x_n, x_{n+1})|^2 \right)^{p/2} dx_1 \cdots dx_n dx_{n+1}, \end{aligned}$$

where in the last step we use the inequality in the preceding alignment. To make this argument precise, we work with finitely many terms and then then pass to limit using Fatou's lemma. \square

Remark 3.4. As a consequence of (3.9) one can derive the following inequality:

$$\|f\|_{L^p(\mathbb{R}^{nl})} \leq C\|f\|_{H^p(\mathbb{R}^l \times \dots \times \mathbb{R}^l)} \quad \text{for } f \in L^2(\mathbb{R}^{nl}), \quad 0 < p \leq 1$$

where $H^p(\underbrace{\mathbb{R}^l \times \dots \times \mathbb{R}^l}_{n \text{ times}})$ denotes the multiparameter Hardy space; on this see [22].

4. THE PROOF OF THEOREM 2.1

Proof. For $1 \leq k \neq \ell \leq m$, we introduce sets

$$U_{k,\ell} = \left\{ (y_1, \dots, y_m) \in (\mathbb{R}^n)^m : \max_{j \neq k, \ell} |y_j| \leq \frac{11}{10}|y_k| \leq \frac{11}{50m}|y_\ell| \right\}$$

and

$$W_{k,\ell} = \left\{ (y_1, \dots, y_m) \in (\mathbb{R}^n)^m : \max_{j \neq k, \ell} |y_j| \leq \frac{11}{10}|y_k|, \frac{1}{10m}|y_\ell| \leq |y_k| \leq 2|y_\ell| \right\}.$$

We now construct smooth homogeneous of degree zero functions $\phi_{k,\ell}$ and $\psi_{k,\ell}$ supported in $U_{k,\ell}$ and $W_{k,\ell}$, respectively, and such that

$$(4.1) \quad \sum_{1 \leq k \neq \ell \leq m} \left(\phi_{k,\ell}(y_1, \dots, y_m) + \psi_{k,\ell}(y_1, \dots, y_m) \right) = 1$$

for every (y_1, \dots, y_m) in $(\mathbb{R}^n)^m \setminus \{0\}$; such functions can be constructed following the hint of Exercise 7.5.4 in [15]. In the support of $\phi_{k,\ell}$ the vector with the largest magnitude is y_ℓ , while in the support of $\psi_{k,\ell}$ the vector with the largest magnitude is y_ℓ and with the second largest one is y_k .

This decomposition of unity leads to a discretization of σ as follows

$$\begin{aligned} \sigma &= \sigma(\xi_{11}, \dots, \xi_{1l}, \dots, \xi_{m1}, \dots, \xi_{ml}) \prod_{j=1}^l \sum_{1 \leq k \neq \ell \leq m} \left(\phi_{k,\ell}(\xi_{1j}, \dots, \xi_{mj}) + \psi_{k,\ell}(\xi_{1j}, \dots, \xi_{mj}) \right) \\ &= \sum \sigma^*(\xi_{11}, \dots, \xi_{1l}, \dots, \xi_{m1}, \dots, \xi_{ml}), \end{aligned}$$

where $\xi_{ij} \in \mathbb{R}^n$. We let $\vec{\xi}_1 = (\xi_{11}, \dots, \xi_{1l})$, $\vec{\xi}_2 = (\xi_{21}, \dots, \xi_{2l})$, etc be elements of \mathbb{R}^{nl} and we notice that these elements appear as inputs of $\widehat{f}_1, \widehat{f}_2$, etc in the definition of $T_\sigma(f_1, \dots, f_m)$. (In this section elements of \mathbb{R}^{nl} are denoted with arrows.) For a given σ^* in the preceding sum we have collections of sets $\{E_i\}_{i=1}^m$ and $\{F_i\}_{i=1}^m$ associated with it that satisfy the following conditions: $\{E_i\}$ is a disjoint partition of $\{1, 2, \dots, l\}$ such that for any $j \in E_i$, $|\xi_{ij}|$ is the largest among $|\xi_{1j}|, |\xi_{2j}|, \dots, |\xi_{mj}|$. Some sets E_i could be empty but they obviously satisfy $\sum_i |E_i| = l$. $\{F_i\}$ is a disjoint collection such that $F_i \cap E_i = \emptyset$ and $F_i \subset \cup_{\kappa} E_\kappa$, moreover, for any $j \in F_i$, $|\xi_{ij}|$ is the second largest among $|\xi_{1j}|, |\xi_{2j}|, \dots, |\xi_{mj}|$. For a given j , either $\phi_{k,\ell}(\xi_{1j}, \dots, \xi_{mj})$ or $\psi_{k,\ell}(\xi_{1j}, \dots, \xi_{mj})$

appears in the product forming σ^* . For a given σ^* we say that the index $j \in \{1, \dots, l\}$ is of case 1 if the term appears $\phi_{k,\ell}(\xi_{1j}, \dots, \xi_{mj})$ in the product and case 2 if the term $\psi_{k,\ell}(\xi_{1j}, \dots, \xi_{mj})$ appears in the product.

For $1 \leq i \leq m$, let $E'_i = E_i \cap (\cup_{\kappa=1}^m F_\kappa)$ and $E''_i = E_i \setminus E'_i$. Then $\cup_{i=1}^m E'_i = \cup_{i=1}^m F_i$ and $E''_i \cap F_i = \emptyset$. If $j \in E'_i$, then $|\xi_{ij}|$ is the largest in case 2, which has the corresponding second largest one $|\xi_{\kappa j}|$ with $j \in F_\kappa$ for some $\kappa \neq i$.

For g_i Schwartz functions on \mathbb{R}^{nl} , we need the following auxiliary estimate

$$(4.2) \quad \|T_{\sigma^*}(g_1, \dots, g_m)\|_{L^2(\mathbb{R}^{nl})} < \infty.$$

To prove (4.2) we argue as follows. We notice that $T_{\sigma^*}(g_1, \dots, g_m)^{\wedge}(\vec{\zeta})$ is equal to

$$\int_{(\mathbb{R}^{ln})^{m-1}} \sigma^*(\vec{\xi}_1, \dots, \vec{\xi}_{m-1}, \vec{\zeta} - (\vec{\xi}_1 + \dots + \vec{\xi}_{m-1})) \prod_{i=1}^{m-1} \widehat{g_i}(\vec{\xi}_i) \widehat{g_m}(\vec{\zeta} - (\vec{\xi}_1 + \dots + \vec{\xi}_{m-1})) d\vec{\xi}_1 \dots d\vec{\xi}_{m-1}.$$

It follows from this, via the Minkowski integral inequality, that the L^2 norm in $\vec{\zeta}$ of the preceding expression is at most

$$\|\sigma\|_{L^\infty(\mathbb{R}^{m ln})} \prod_{i=1}^{m-1} \|\widehat{g_i}\|_{L^1(\mathbb{R}^{ln})} \|g_m\|_{L^2(\mathbb{R}^{ln})} < \infty$$

and this proves (4.2). Thus, for g_1, \dots, g_m Schwartz functions on \mathbb{R}^{ln} (4.2) holds.

For functions with m variables in \mathbb{R}^n , let \mathcal{F}_k denote the Fourier transform in the k th variable and by \mathcal{F}_k^{-1} the inverse Fourier transform. For an appropriate smooth bump $\widehat{\theta}$ which is supported in an annulus in \mathbb{R}^n , we define an operator

$$\Delta_j^{(k)}(f)(x_1, \dots, x_{k-1}, x_k, x_{k+1}, x_m) = \mathcal{F}_k^{-1} \left(\widehat{\theta}(2^{-j}\xi) \mathcal{F}_k(f)(x_1, \dots, x_{k-1}, \xi, x_{k+1}, x_m) \right) (x_k).$$

For $F = \{k_1, k_2, \dots, k_\rho\}$, we define the expression $\sum_{J_F} \Delta_{J_F}$ as follows:

$$\sum_{J_F} \Delta_{J_F}(f) = \sum_{j_{k_1}} \dots \sum_{j_{k_\rho}} \Delta_{j_{k_1}}^{(k_1)} \dots \Delta_{j_{k_\rho}}^{(k_\rho)}(f).$$

Using the fact that $\cup_i E'_i = \cup_i F_i$, we write

$$\begin{aligned} T_{\sigma^*}(f_1, \dots, f_m) &= \sum_{J_{F_1}} \dots \sum_{J_{F_m}} T_{\sigma^*}(\Delta_{J_{F_1}} f_1, \Delta_{J_{F_2}} f_2, \dots, \Delta_{J_{F_m}} f_m) \\ &= \sum_{J_{F_1}} \dots \sum_{J_{F_m}} T_{\sigma^*}(\Delta_{J_{E'_1 \cup F_1}} f_1, \Delta_{J_{E'_2 \cup F_2}} f_2, \dots, \Delta_{J_{E'_m \cup F_m}} f_m), \end{aligned}$$

where $\Delta_{J_{E'_i \cup F_i}}$ are associated with a bump which is equal to one on the support of $\widehat{\theta}$.

Recall our assumptions that $1 < r \leq 2$, $\frac{1}{r} < \bar{\gamma}_{ij}$ and $\frac{1}{p_i} < \bar{\gamma}_{ij}$ for $1 \leq i \leq m$, $1 \leq j \leq l$. Pick $\rho_i \in (1, 2)$ such that $\min(p_i, r) > \rho_i$ and $\rho_i \bar{\gamma}_{ij} > 1$ for all $1 \leq i \leq m$, $1 \leq j \leq l$.

Then for any $p > 0$, using that $T_{\sigma^*}(f_1, \dots, f_m) \in L^2(\mathbb{R}^{nl})$, Lemma 3.3 yields

$$\begin{aligned} & \|T_{\sigma^*}(f_1, \dots, f_m)\|_{L^p(\mathbb{R}^{ln})} \\ & \lesssim \left\| \left[\sum_{J_{E'_1}''} \cdots \sum_{J_{E'_m}''} \left| \Delta_{J_{E'_1}''} \cdots \Delta_{J_{E'_m}''} \left\{ \sum_{J_{F_1}} \cdots \sum_{J_{F_m}} T_{\sigma^*}(\Delta_{J_{E'_1}'' \cup F_1} f_1, \dots, \Delta_{J_{E'_m}'' \cup F_m} f_m) \right\} \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{ln})} \\ & \lesssim \left\| \left[\sum_{J_{E'_1}''} \cdots \sum_{J_{E'_m}''} \left| \sum_{J_{F_1}} \cdots \sum_{J_{F_m}} \Delta_{J_{E'_1}''} \cdots \Delta_{J_{E'_m}''} T_{\sigma^*}(\Delta_{J_{E'_1}'' \cup E'_1 \cup F_1} f_1, \dots, \Delta_{J_{E'_m}'' \cup E'_m \cup F_m} f_m) \right|^2 \right]^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{ln})}. \end{aligned}$$

Using Lemmas 4.1 and 4.2 stated below we estimate the last displayed expression by

$$\begin{aligned} & \left\| \left(\sum_{J_{E'_1}''} \cdots \sum_{J_{E'_m}''} \left\{ \sum_{J_{F_1}} \cdots \sum_{J_{F_m}} \left(\prod_{k=1}^m M_{\rho_k}(\Delta_{J_{E'_k}'' \cup E'_k \cup F_k} f_k) \right) \right\}^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{ln})} \\ & \lesssim \left\| \left(\prod_{k=1}^m \left[\sum_{J_{E'_k}''} \sum_{J_{E'_k}'} \sum_{J_{F_k}} M_{\rho_k}(\Delta_{J_{E'_k}'' \cup E'_k \cup F_k} f_k)^2 \right] \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^{ln})} \\ & \lesssim \prod_{k=1}^m \left\| \left(\sum_{J_{E'_k}''} \sum_{J_{E'_k}'} \sum_{J_{F_k}} |\Delta_{J_{E'_k}'' \cup E'_k \cup F_k} f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^{p_k}(\mathbb{R}^{ln})} \lesssim \prod_{k=1}^m \|f_k\|_{L^{p_k}(\mathbb{R}^{ln})}. \end{aligned}$$

Lemma 4.1. Denote $\Delta_{J_{E'_i}'' \cup E'_i \cup F_i}(f_i)$ by \tilde{f}_i for $i = 1, \dots, m$, then we have

$$|\Delta_{J_{E'_1}''} \cdots \Delta_{J_{E'_m}''}(T_{\sigma^*}(\tilde{f}_1, \dots, \tilde{f}_m))| \lesssim A M_{\rho_1}(\tilde{f}_1) \cdots M_{\rho_m}(\tilde{f}_m).$$

Proof. We prove this lemma only in the case $\rho_1 = \cdots = \rho_m = \rho$ which is enough for the proof of Theorem 1.1 where we have $\gamma_{ijk} = \gamma_0$ for some fixed γ_0 . Thus the idea of the proof is given without introducing even more cumbersome notation. We note that for the case where the ρ_i are different we additionally need Minkowski's integral inequality.

For $\vec{x} = (x_1, \dots, x_l) \in \mathbb{R}^{nl}$, we define $D_J(x) = (2^{j_k \chi_E(k)} x_k)_{k=1}^l$ with $E = \cup_i E_i''$, which means that if $k \in E$, then we dilate the k -th coordinate (which is a vector in \mathbb{R}^n) by 2^{j_k} . We define $D_{-J}(\vec{x}) = (2^{-j_k \chi_E(k)} x_k)_{k=1}^l$ and also define:

$$D_J(\vec{x} - \vec{y}_1, \dots, \vec{x} - \vec{y}_m) = (D_J(\vec{x} - \vec{y}_1), \dots, D_J(\vec{x} - \vec{y}_m)).$$

Finally, for a function G on $(\mathbb{R}^{nl})^m$ we define $G_J = G \circ D_J$.

Let us define

$$\widehat{\Theta}(\vec{\xi}_1, \dots, \vec{\xi}_m) = \prod_{j \in E} \widehat{\theta}(\xi_{1j} + \xi_{2j} + \cdots + \xi_{mj}),$$

then $\Delta_{J_{E_1''}} \cdots \Delta_{J_{E_m''}} (T_{\sigma^*}(\tilde{f}_1, \dots, \tilde{f}_m))(\vec{x})$ is equal to

$$\int_{\mathbb{R}^{mln}} (\sigma^* \widehat{\Theta}_{-J})(\vec{\xi}_1, \dots, \vec{\xi}_m) \tilde{f}_1(\vec{\xi}_1) \cdots \tilde{f}_m(\vec{\xi}_m) e^{2\pi i \vec{x} \cdot (\vec{\xi}_1 + \cdots + \vec{\xi}_m)} d\vec{\xi}_1 \cdots d\vec{\xi}_m.$$

In the support of this integral, we have that $|\xi_{1\ell}| + \cdots + |\xi_{m\ell}| \sim 2^{j_\ell}$ for $\ell = 1, \dots, l$. In this case, one may insert the term

$$\widehat{\Psi}_*(D_{-J}(\vec{\xi}_1, \dots, \vec{\xi}_m))$$

where

$$(4.3) \quad \widehat{\Psi}_*(\vec{\xi}_1, \dots, \vec{\xi}_m) = \prod_{\ell=1}^l \left[\sum_{|k| \leq 10} \widehat{\Psi}(2^{-k}(\xi_{1\ell} + \cdots + \xi_{m\ell})) \right]$$

in the integrand, where $\widehat{\Psi}$ is the function in (2.7).

Let $j_{E_i''} = \sum_{k \in E_i''} j_k$. Taking the inverse Fourier transform, then we can rewrite $\Delta_{J_{E_1''}} \cdots \Delta_{J_{E_m''}} T_{\sigma^*}(\tilde{f}_1, \dots, \tilde{f}_m)(\vec{x})$ as

$$(4.4) \quad \int_{(\mathbb{R}^{ln})^m} 2^{mn(j_{E_1''} + \cdots + j_{E_m''})} (\sigma_J^* \widehat{\Psi}_* \widehat{\Theta})^\vee(D_J(\vec{x} - \vec{y}_1, \dots, \vec{x} - \vec{y}_m)) \prod_{i=1}^m \tilde{f}_i(\vec{y}_i) d\vec{y}_1 \cdots d\vec{y}_m,$$

where $\vec{y}_1, \dots, \vec{y}_m \in \mathbb{R}^{ln}$. Define $M_\rho(f) = M(|f|^\rho)^{1/\rho}$, where M is the strong maximal function on \mathbb{R}^{nl} . Insert in the previous integral the term

$$\frac{\prod_{i=1}^m \omega_{\vec{\gamma}_i}(D_J(\vec{x} - \vec{y}_i))}{\prod_{i=1}^m \omega_{\vec{\gamma}_i}(D_J(\vec{x} - \vec{y}_i))},$$

use the fact that, for $x_1 \in \mathbb{R}^n$ and $\vec{\gamma} > 1$,

$$\int_{\mathbb{R}^n} \frac{2^j |g(y)|}{\omega_{\vec{\gamma}}(2^j(x_1 - y))} dy \lesssim M(g)(x_1),$$

and apply Hölder's inequality with respect to the indices ρ and ρ' . Then we control (4.4) by

$$\left(\int_{(\mathbb{R}^{ln})^m} 2^{mn \sum_i j_{E_i''}} |(\sigma_J^* \widehat{\Psi}_* \widehat{\Theta})^\vee(D_J(\vec{y}_1, \dots, \vec{y}_m)) \prod_{i=1}^m \omega_{\vec{\gamma}_i}(D_J(\vec{y}_i))|^{\rho'} d\vec{y}_1 \cdots d\vec{y}_m \right)^{\frac{1}{\rho'}} \prod_{i=1}^m M_\rho(\tilde{f}_i)(\vec{y}_i).$$

Changing variables in the preceding integral, we write it as $\|(\sigma_J^* \widehat{\Psi}_* \widehat{\Theta})^\vee \omega_{\vec{\gamma}}\|_{L^{\rho'}(\mathbb{R}^{nl})^m}$, which by (3.3) is controlled as follows,

$$\|(\sigma_J^* \widehat{\Psi}_* \widehat{\Theta})^\vee \omega_{\vec{\gamma}}\|_{L^{\rho'}((\mathbb{R}^{nl})^m)} \leq C \|(\sigma_J^* \widehat{\Psi}_* \widehat{\Theta})^\vee \omega_{\vec{\gamma}}\|_{L^{\rho'}((\mathbb{R}^{nl})^m)} \leq C \|\sigma_J^* \widehat{\Psi}_* \widehat{\Theta}\|_{L_{\vec{\gamma}}^r((\mathbb{R}^{nl})^m)}.$$

Notice that $\phi_{k,\ell}$ and $\psi_{k,\ell}$ are smooth and homogeneous, then by (3.4), the last norm is less than $\|\sigma_J \widehat{\Psi}_*\|_{L_{\vec{\gamma}}^r}$, which is smaller than A by assumption and Lemma 3.2. \square

Lemma 4.2. *Under the preceding notation, we have*

$$\sum_{J_{F_1}} \cdots \sum_{J_{F_m}} \left[\prod_{k=1}^m M_{\rho_k}(\Delta_{J_{E'_k \cup E'_k \cup F_k}} f_k) \right] \leq \left(\prod_{k=1}^m \left[\sum_{J_{E'_k}} \sum_{J_{F_k}} M_{\rho_k}(\Delta_{J_{E'_k \cup E'_k \cup F_k}} f_k)^2 \right] \right)^{\frac{1}{2}}.$$

Proof. This is a series of applications of the Cauchy-Schwarz inequality. We define $E'_{ij} = E'_i \cap F_j$. Notice that $E'_{ii} = E'_i \cap F_i = \emptyset$. Because $\cup_j F_j = \cup_i E'_i$, we have $E'_i = \cup_j E'_{ij}$ and $F_j = \cup_i E'_{ij}$. Then

$$\begin{aligned} & \sum_{J_{F_1}} \cdots \sum_{J_{F_m}} \left[\prod_{k=1}^m M_{\rho_k}(\Delta_{J_{E'_k \cup E'_k \cup F_k}} f_k) \right] \\ &= \sum_{i,j=1}^m \sum_{J_{E'_{ij}}} \left(\prod_{k=1}^m M_{\rho_k}(\Delta_{J_{E'_k \cup (\cup_j E'_{kj}) \cup (\cup_i E'_{ik})}} f_k) \right) \\ &\leq \left(\prod_{k=1}^m \left[\sum_j \sum_{J_{E'_{kj}}} \sum_i \sum_{J_{E'_{ik}}} M_{\rho_k}(\Delta_{J_{E'_k \cup (\cup_j E'_{kj}) \cup (\cup_i E'_{ik})}} f_k)^2 \right] \right)^{\frac{1}{2}} \\ &= \left(\prod_{k=1}^m \left[\sum_{J_{E'_k}} \sum_{J_{F_k}} M_{\rho_k}(\Delta_{J_{E'_k \cup E'_k \cup F_k}} f_k)^2 \right] \right)^{\frac{1}{2}}. \end{aligned}$$

Lemma 4.2 is now proved. \square

This completes the proof of Theorem 2.1. \square

5. THE CASE $n = k = 1$ OF THEOREM 1.1

In this section we prove the case $n = k = 1$ of Theorem 1.1 and we also provide some essential lemmas needed for estimating the local Sobolev norm of the symbols $\sigma_k^{(1)}$ for all $k \geq 1$. For simplicity, we use the variables (ξ, η) instead of (ξ_0, ξ_1) in $\sigma_1^{(1)}(\xi_0, \xi_1)$.

We need the following characterizations of Sobolev norms, given by Stein [37], [38, Lemma 3, p. 136].

Lemma 5.1 (Stein). *(i) Let $0 < \alpha < 1$ and $2n/(n + 2\alpha) < p < \infty$. Then $f \in L_\alpha^p(\mathbb{R}^n)$ if and only if $\|f\|_{L_\alpha^p(\mathbb{R}^n)} \simeq \|f\|_{L^p(\mathbb{R}^n)} + \|I_\alpha(f)\|_{L^p(\mathbb{R}^n)}$ where*

$$I_\alpha(f)(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+2\alpha}} dy \right)^{1/2}.$$

(ii) Let $1 \leq \alpha < \infty$ and $1 < p < \infty$. Then $f \in L_\alpha^p(\mathbb{R}^n)$ if and only if $f \in L_{\alpha-1}^p(\mathbb{R}^n)$ and for $1 \leq j \leq n$, $\frac{\partial f}{\partial x_j} \in L_{\alpha-1}^p(\mathbb{R}^n)$. Furthermore, we have

$$\|f\|_{L_\alpha^p(\mathbb{R}^n)} \simeq \|f\|_{L_{\alpha-1}^p(\mathbb{R}^n)} + \sum_{j=1}^n \left\| \frac{\partial f}{\partial x_j} \right\|_{L_{\alpha-1}^p(\mathbb{R}^n)}.$$

Throughout this section fix a nondecreasing smooth function h on \mathbb{R} such that

$$(5.1) \quad h(t) = \begin{cases} 3, & \text{if } t \in [4, +\infty); \\ \text{smooth}, & \text{if } t \in [2, 4); \\ t, & \text{if } t \in [1/8, 2); \\ \text{smooth}, & \text{if } t \in [1/32, 1/8); \\ 1/16, & \text{otherwise.} \end{cases}$$

Lemma 5.2. *Let u be a function supported in the rectangle*

$$(5.2) \quad \{(y_1, y_2) : |y_1| \leq 101/100, 1/4 \leq y_2 \leq 7/4\}$$

in \mathbb{R}^2 such that $\nabla u \in L^\infty(\mathbb{R}^2)$, and $u(x) \in L_\gamma^r(\mathbb{R}^2)$ with $1 < \gamma < 2$, $2/\gamma < r < 1/(\gamma - 1)$. Define $U(y_1, y_2) = u(y_1/h(y_2), y_2)$. Then $U \in L_\gamma^r(\mathbb{R}^2)$ and

$$\|U\|_{L_\gamma^r(\mathbb{R}^2)} \leq C(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L_\gamma^r(\mathbb{R}^2)}).$$

Proof. Because of Lemma 5.1, it suffices to show for $\alpha = \gamma - 1$ and $2/(1 + \alpha) < r < 1/\alpha$ that $U \in L_1^r(\mathbb{R}^2)$, $I_\alpha(U) \in L^r(\mathbb{R}^2)$ and $I_\alpha(\partial_j U) \in L^r(\mathbb{R}^2)$ with $j = 1, 2$. The first assertion follows trivially by checking the derivatives directly while the second one is verified in a way similar to the third one, where we adapt an argument found in Triebel [40, Section 4.3] with a suitable change of variables.

Next, we show that $I_\alpha(\partial_1 U) \in L^r(\mathbb{R}^2)$. We will estimate the following expression

$$\|I_\alpha(\partial_1 U)\|_{L^r(\mathbb{R}^2)}^r = \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{|y - y'|^{2+2\alpha}} dy \right)^{r/2} dy'.$$

Denote by B a finite ball centered at 0 containing the support of $\partial_1 U$. Then it is easy to check that, since $\partial_1 U \in L^\infty$, $r(1 + \alpha) = r\gamma > 2$,

$$\|I_\alpha(\partial_1 U)\|_{L^r(\mathbb{R}^2)}^r \leq C \left(\|\nabla u\|_{L^\infty}^r + \int_{3B} \left(\int_{3B} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{|y - y'|^{2+2\alpha}} dy \right)^{r/2} dy' \right),$$

where C is a constant depending on B .

Denote $x = (x_1, x_2)$, $y = (y_1, y_2)$. One writes $y = \varphi(x)$ and $x = \psi(y)$ in the form

$$\begin{cases} y_1 &= \varphi_1(x_1, x_2) = x_1 h(x_2), \\ y_2 &= \varphi_2(x_1, x_2) = x_2 \end{cases}$$

and

$$\begin{cases} x_1 &= \psi_1(y_1, y_2) = y_1/h(y_2), \\ x_2 &= \psi_2(y_1, y_2) = y_2, \end{cases}$$

where h is a function defined in (5.1). By the change of variables $y = \varphi(x)$ with $|\det \varphi'(x)| < C < \infty$, direct computations give

$$\begin{aligned} \partial_1 U(y) &= \frac{\partial}{\partial y_1} u(\psi(y)) \cdot \frac{1}{h(y_2)} =: \partial_1 u(\psi(y)) \cdot \frac{1}{h(y_2)}, \\ \partial_2 U(y) &= -\frac{\partial}{\partial y_1} u(\psi(y)) \cdot \frac{y_1 h'(y_2)}{h(y_2)} + \frac{\partial}{\partial y_2} u(\psi(y)) =: -\partial_1 u(\psi(y)) \cdot \frac{y_1 h'(y_2)}{h(y_2)} + \partial_2 u(\psi(y)), \end{aligned}$$

and the fact that $|\psi(y) - \psi(y')| \leq \max\{\|\nabla \psi_1\|_\infty, \|\nabla \psi_2\|_\infty\}|y - y'|$, we have

$$\begin{aligned} &\|I_\alpha(\partial_1 U)\|_{L^r(\mathbb{R}^2)}^r \\ &\leq C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|\partial_1 U(y) - \partial_1 U(y')|^2}{|\psi(y) - \psi(y')|^{2+2\alpha}} dy \right)^{r/2} dy' \\ &\leq C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{\left| \frac{\partial_1 u(\psi(y))}{h(y_2)} - \frac{\partial_1 u(\psi(y'))}{h(y'_2)} \right|^2}{|\psi(y) - \psi(y')|^{2+2\alpha}} dy \right]^{r/2} dy' \\ &\leq C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{\left| \frac{\partial_1 u(x)}{h(x_2)} - \frac{\partial_1 u(x')}{h(x'_2)} \right|^2}{|x - x'|^{2+2\alpha}} |\det \varphi'(x)| dx \right]^{r/2} |\det \varphi'(x')| dx' \\ &\leq C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C \int_{\mathbb{R}^2} \left[\int_{\mathbb{R}^2} \frac{\left| \frac{\partial_1 u(x)}{h(x_2)} - \frac{\partial_1 u(x')}{h(x'_2)} \right|^2}{|x - x'|^{2+2\alpha}} dx \right]^{r/2} dx'. \end{aligned}$$

Now take $\eta(x_1, x_2) \in C_0^\infty(\mathbb{R}^2)$ assuming value 1 on the support of $\partial_1 u$ so that the support of η is just a bit larger than that of $\partial_1 u$, and $h(x_2) = x_2$ on the support of η . Define $\tilde{h}(x_1, x_2) = \eta(x_1, x_2)/h(x_2)$ and then write

$$\begin{aligned} \frac{\partial_1 u(x)}{h(x_2)} - \frac{\partial_1 u(x')}{h(x'_2)} &= \partial_1 u(x) \tilde{h}(x) - \partial_1 u(x') \tilde{h}(x') \\ &= [\partial_1 u(x) - \partial_1 u(x')] \tilde{h}(x') + \partial_1 u(x) [\tilde{h}(x) - \tilde{h}(x')], \end{aligned}$$

which yields

$$\begin{aligned} \|I_\alpha(\partial_1 U)\|_{L^r(\mathbb{R}^2)}^r &\leq C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|\partial_1 u(x) - \partial_1 u(x')|^2}{|x - x'|^{2+2\alpha}} dx \right)^{r/2} dx' \\ &\quad + C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|\tilde{h}(x) - \tilde{h}(x')|^2}{|x - x'|^{2+2\alpha}} dx_1 dx_2 \right)^{r/2} dx'_1 dx'_2 \\ &\leq C\|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C\|\partial_1 u\|_{L^\infty_\alpha(\mathbb{R}^2)}^r + C\|\nabla u\|_{L^\infty}^r \|\tilde{h}\|_{L^\infty_\alpha(\mathbb{R}^2)}^r \end{aligned}$$

$$\leq C \left(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L_\gamma^r(\mathbb{R}^2)} \right)^r.$$

A similar argument as the one above shows that

$$\begin{aligned} \|I_\alpha(\partial_2 U)\|_{L^r(\mathbb{R}^2)}^r &= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|\partial_2 U(y) - \partial_2 U(y')|^2}{|y - y'|^{2+2\alpha}} dy \right)^{r/2} dy' \\ &\leq C \|\nabla u\|_{L^\infty(\mathbb{R}^2)}^r + C \|\partial_1 u\|_{L_\alpha^r(\mathbb{R}^2)}^r + C \|\partial_2 u\|_{L_\alpha^r(\mathbb{R}^2)}^r \\ &\leq C \left(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L_\gamma^r(\mathbb{R}^2)} \right)^r. \end{aligned}$$

Also, by repeating the preceding argument we obtain,

$$\|I_\alpha(U)\|_{L^r(\mathbb{R}^2)} \leq C \left(\|u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L_\alpha^r(\mathbb{R}^2)} \right) \leq C \|u\|_{L_\gamma^r(\mathbb{R}^2)},$$

where we used the Sobolev embedding theorem in the last inequality with $\gamma r > 2$. The proof of Lemma 5.2 is now complete. \square

For g, h on \mathbb{R} define a the tensor $g \otimes h$ as the following function on \mathbb{R}^2 by setting $(g \otimes h)(\xi, \eta) = g(\xi)h(\eta)$.

Lemma 5.3. *Let $f \in L_\gamma^r(\mathbb{R})$ supported in $[-1, 1]$, and $\widehat{\Theta}$ is a smooth function supported in an annulus centered at 0 with size comparable to 1, then we have*

$$\|f \otimes \widehat{\Theta}\|_{L_\gamma^r(\mathbb{R}^2)} \leq C \|f\|_{L_\gamma^r(\mathbb{R})}.$$

Proof. We use the same idea as in the proof of Lemma 5.2. It suffices to prove that $f \otimes \widehat{\Theta} \in L_1^r(\mathbb{R}^2)$ and that $I_\alpha(\partial^\beta(f \otimes \widehat{\Theta})) \in L^r(\mathbb{R}^2)$ with $|\beta| = 1$. It is easy to check that $\|f \otimes \widehat{\Theta}\|_{L_1^r} \leq C \|f\|_{L_1^r}$, so we only prove that $I_\alpha(\partial_\xi(f \otimes \widehat{\Theta})) \in L^r(\mathbb{R}^2)$.

Note that $f \otimes \widehat{\Theta}$ is compactly supported and we can choose a function $\varphi(\xi, \eta) \in C_0^\infty(\mathbb{R}^2)$ assuming 1 on the support of $f \otimes \widehat{\Theta}$ and therefore $f \otimes \widehat{\Theta} = f(\xi)\varphi(\xi, \eta)\widehat{\Theta}(\eta)\varphi(\xi, \eta)$. Then $\int_{\mathbb{R}^2} |I_\alpha(\partial_\xi(f \otimes \widehat{\Theta}))|^r d\xi d\eta$ is split into the parts

$$\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|[f'(\xi)\varphi(\xi, \eta) - f'(\xi')\varphi(\xi', \eta')]\widehat{\Theta}(\eta')\varphi(\xi', \eta')|^2}{|(\xi, \eta) - (\xi', \eta')|^{2+2\alpha}} d\xi' d\eta' \right)^{r/2} d\xi d\eta$$

and

$$\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \frac{|f'(\xi)\varphi(\xi, \eta)[\widehat{\Theta}(\eta)\varphi(\xi, \eta) - \widehat{\Theta}(\eta')\varphi(\xi', \eta')]|^2}{|(\xi, \eta) - (\xi', \eta')|^{2+2\alpha}} d\xi' d\eta' \right)^{r/2} d\xi d\eta.$$

We prove only that the first one is finite since the latter can be proved similarly.

To prove the boundedness of the first one, we split it further via the identity

$$f'(\xi)\varphi(\xi, \eta) - f'(\xi')\varphi(\xi', \eta') = (f'(\xi) - f'(\xi'))\varphi(\xi, \eta) + f'(\xi')(\varphi(\xi, \eta) - \varphi(\xi', \eta')).$$

The integral containing the second part is finite because f' is bounded and $\varphi \in L_\gamma^r(\mathbb{R}^2)$. For the other part, a simple change of variable $\eta' \rightarrow (\eta - \eta')/(\xi - \xi')$ shows that it is equal to

$$C \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} \frac{|f'(\xi) - f'(\xi')|^2}{|\xi - \xi'|^{1+2\alpha}} d\xi' \right)^{r/2} |\varphi(\xi, \eta)| d\xi d\eta,$$

which, by Lemma 5.1, is bounded by $\|f\|_{L_\gamma^r(\mathbb{R})}^r$ since $\varphi \in C_0^\infty(\mathbb{R}^2)$. \square

Lemma 5.4. *Let $\gamma \in (1, 2)$ and $1 < r < \frac{1}{\gamma-1}$. Then $\|\Phi\varphi\|_{L_\gamma^r(\mathbb{R})} < \infty$, where φ is a smooth function with compact support, and Φ is the function in (1.7).*

Proof. To obtain the claim, we need to show that $J^\gamma(\varphi\Phi) = ((1 + |\xi|^2)^{\gamma/2} \widehat{\varphi\Phi})^\vee \in L^r(\mathbb{R})$. Since

$$\|J^\gamma(\varphi\Phi)\|_{L^r(\mathbb{R})} \approx \|\varphi\Phi\|_{L^r(\mathbb{R})} + \|D_\xi^\gamma(\varphi\Phi)\|_{L^r(\mathbb{R})},$$

and trivially $\varphi\Phi \in L^r(\mathbb{R})$, we reduce the proof to establishing $\|D_\xi^\gamma(\varphi\Phi)\|_{L^r(\mathbb{R})} < \infty$, where $D_\xi^\gamma(\varphi\Phi) = (|\xi|^\gamma \widehat{\varphi\Phi})^\vee$. By the Kato-Ponce inequality for homogeneous type [7], [34], [19], it suffices to show that $D_\xi^\gamma(\Phi)$ lies in $L^r(\mathbb{R})$. Indeed, for $\gamma \in (1, 2)$ we write

$$\begin{aligned} \widehat{\Phi}(\xi) |\xi|^\gamma &= \frac{1}{\xi} \xi \widehat{\Phi}(\xi) |\xi|^\gamma = \frac{1}{2\pi i} \frac{1}{\xi} \widehat{\Phi}'(\xi) |\xi|^\gamma \\ &= -i \frac{1}{\pi \xi} \widehat{\chi_{[-1,0]}}(\xi) |\xi|^\gamma = -i \frac{1}{\pi \xi} \frac{e^{2\pi i \xi} - 1}{2\pi i \xi} |\xi|^\gamma \\ &= -i \frac{1}{\pi} \frac{e^{2\pi i \xi} - 1}{2\pi i} |\xi|^{\gamma-2} = -\frac{1}{2\pi^2} (e^{2\pi i \xi} - 1) |\xi|^{\gamma-2}. \end{aligned}$$

Taking inverse Fourier transforms we obtain that

$$(\widehat{\Phi}(\xi) |\xi|^\gamma)^\vee(x) = c_\gamma (|x+1|^{1-\gamma} - |x|^{1-\gamma})$$

and this function lies in $L^r(\mathbb{R})$ when $1 < r < \frac{1}{\gamma-1}$ and γ is very close to 2. \square

The preceding result can be lifted to \mathbb{R}^2 as follows.

Lemma 5.5. *Let $\gamma \in (1, 2)$ and $1 < r < \frac{1}{\gamma-1}$, and let θ be a function supported in $\frac{1}{2} \leq |\xi| \leq 2$ on the real line. Define $U(\xi, \eta) = \Phi(\frac{\xi}{\eta}) \theta(\frac{\xi}{\eta}) \widehat{\psi}(\xi, \eta)$, where $\widehat{\psi}$ is a smooth function supported in an annulus centered at zero. Then $\|U\|_{L_\gamma^r(\mathbb{R}^2)} < \infty$.*

Proof. Set

$$u(\xi, \eta) = \Phi(\xi) \theta(\xi) \widehat{\Psi}(\xi, \eta)$$

and

$$U(\xi, \eta) = \Phi(\xi/\eta) \theta(\xi/\eta) \widehat{\Psi}(\xi, \eta).$$

Since $h(\eta) = \eta$ on the support of the function U . We now apply Lemma 5.2 to obtain

$$\|U\|_{L_\gamma^r(\mathbb{R}^2)} \leq C(\|\nabla u\|_{L^\infty(\mathbb{R}^2)} + \|u\|_{L_\gamma^r(\mathbb{R}^2)}).$$

Thus, it is enough to show that $\|u\|_{L_\gamma^r(\mathbb{R}^2)} < \infty$. We introduce a compactly supported smooth function $\widehat{\Theta}(\eta)$ which is equal to 1 on the support of $\eta \mapsto \theta(\xi)\widehat{\Psi}(\xi\eta, \eta)$ for any ξ . the Kato-Ponce inequality ([27] [19]) allows us to estimate the Sobolev norm of u as follows:

$$\begin{aligned} \|u\|_{L_\gamma^r(\mathbb{R}^2)} &= \|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\widehat{\Psi}(\xi\eta, \eta)\|_{L_\gamma^r(\mathbb{R}^2)} \\ &\lesssim \|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\|_{L_\gamma^r(\mathbb{R}^2)} \|\widehat{\Psi}(\xi\eta, \eta)\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + \|\widehat{\Psi}(\xi\eta, \eta)\|_{L_\gamma^r(\mathbb{R}^2)} \|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\|_{L^\infty(\mathbb{R}^2)}. \end{aligned}$$

We are left with establishing $\|\Phi(\xi)\theta(\xi)\widehat{\Theta}(\eta)\|_{L_\gamma^r(\mathbb{R}^2)} < \infty$, since all other terms on the right of the above inequality are finite. This is achieved via Lemmas 5.4 and 5.3. Thus the proof of Lemma 5.5 is complete. \square

Using these ideas we are able to prove Theorem 1.1 in the case where $n = k = 1$.

Proposition 5.6. *The Calderón commutator $\mathcal{C}_1^{(1)}$ maps $L^{p_1}(\mathbb{R}) \times L^{p_2}(\mathbb{R})$ to $L^p(\mathbb{R})$ when $1/p_1 + 1/p_2 = 1/p$, $1 < p_1, p_2 < \infty$, and $1/2 < p < \infty$.*

Proof. Note that $\text{sgn}(\eta)\Phi(\xi/\eta)$ has an obvious modification which is continuous on $\mathbb{R}^2 \setminus \{0\}$. We denote the latter by $\text{sgn}(\eta)\Phi(\xi/\eta)$ as well since there is no chance to introduce any confusion.

We introduce a smooth function with compact support θ on the real line which is supported in two small intervals, say, of length $1/100$ centered at the points -1 and 0 . Then we write

$$1 = \theta(\xi/\eta) + 1 - \theta(\xi/\eta)$$

and we decompose the function $\text{sgn}(\eta)\Phi(\xi/\eta) = \sigma_1(\xi, \eta) + \sigma_2(\xi, \eta)$, where $\sigma_1(\xi, \eta) = \text{sgn}(\eta)\Phi(\xi/\eta)\theta(\xi/\eta)$ and $\sigma_2(\xi, \eta) = \text{sgn}(\eta)\Phi(\xi/\eta)(1 - \theta(\xi/\eta))$. Let $\widehat{\Psi}$ be a smooth bump supported in the annulus $1/2 < |(\xi, \eta)| < 3/2$ in \mathbb{R}^2 . The function σ_2 is smooth away from zero and $\sigma_2\widehat{\Psi}$ lies in $L_\gamma^r(\mathbb{R}^2)$ for any $r, \gamma > 1$. Also, $\sigma_1\widehat{\Psi}$ lies in $L_\gamma^r(\mathbb{R}^2)$ with $r\gamma > 1$. in view of Lemma 5.5. Then Corollary 2.2 implies the required conclusion. \square

6. PROOF OF THEOREM 1.1: THE GENERAL CASE

For a function h on \mathbb{R} , we set $J^{\gamma/2}(h) = \mathcal{F}^{-1}((1 + |\xi|^2)^{\gamma/4} \mathcal{F}(h))$, where \mathcal{F} is the Fourier transform. If g has several variables, we set $J_l^{\gamma/2}(g) = \mathcal{F}_l^{-1}((1 + |\xi_l|^2)^{\gamma/4} \mathcal{F}_l(g))$, where \mathcal{F}_l denotes the Fourier transform of g taken in the l th variable with the remaining variables being fixed. We begin with a simple but useful lemma.

Lemma 6.1. *If g_2, \dots, g_{k+1} lie in $L_{\gamma/2, \gamma/2}^r(\mathbb{R}^2)$ and $r\gamma/2 > 1$, then the function*

$$(x_1, \dots, x_k) \mapsto g_2(x_1, x_2) \cdots g_{k+1}(x_1, x_k)$$

lies in $L_{\gamma/2, \dots, \gamma/2}^r(\mathbb{R}^{k+1})$, where $\gamma/2$ appears $k+1$ times. Moreover, the function

$$(6.1) \quad g_2(x_1, x_2) \cdots g_{k+1}(x_1, x_k) \Psi(x_1, \dots, x_{k+1}) \in L_{\gamma/2, \dots, \gamma/2}^r(\mathbb{R}^{k+1})$$

if $\Psi \in \mathcal{S}(\mathbb{R}^{k+1})$, where \mathcal{S} is the space of Schwartz functions.

Proof. We begin by observing that since $r\gamma/2 > 1$, in view of the Sobolev embedding theorem, we have that $\|h\|_{L^\infty(\mathbb{R})} \leq \|J_1^{\gamma/2}(h)\|_{L^r(\mathbb{R})}$. Combining this observation with the Kato-Ponce inequality for inhomogeneous spaces [27], [19] we obtain

$$\|J_1^{\gamma/2}(h_2 h_3)\|_{L^r} \lesssim \|J_1^{\gamma/2}(h_2)\|_{L^r} \|h_3\|_{L^\infty} + \|J_1^{\gamma/2}(h_3)\|_{L^r} \|h_2\|_{L^\infty} \lesssim \|J_1^{\gamma/2}(h_2)\|_{L^r} \|J_1^{\gamma/2}(h_3)\|_{L^r}$$

where all norms are taken in the first variable. By induction it follows that

$$\|J_1^{\gamma/2}(h_2 \cdots h_{k+1})\|_{L^r(\mathbb{R})} \lesssim \|J_1^{\gamma/2}(h_2)\|_{L^r(\mathbb{R})} \cdots \|J_1^{\gamma/2}(h_{k+1})\|_{L^r(\mathbb{R})}.$$

Using this inequality, we obtain

$$\begin{aligned} \|g_2 \cdots g_{k+1}\|_{L_{\vec{\gamma}}^r(\mathbb{R}^{k+1})} &= \left\| \|J_1^{\gamma/2}(J_2^{\gamma/2}(g_2) \cdots J_{k+1}^{\gamma/2}(g_{k+1}))\|_{L^r(x_1)} \right\|_{L^r(x_2, \dots, x_{k+1})} \\ &\lesssim \left\| \prod_{l=2}^{k+1} \|J_1^{\gamma/2} J_l^{\gamma/2}(g_l)\|_{L^r(x_1)} \right\|_{L^r(x_2, \dots, x_{k+1})} \\ &= C \prod_{l=2}^{k+1} \|J_1^{\gamma/2} J_l^{\gamma/2}(g_l)\|_{L^r(x_1, x_l)} < \infty. \end{aligned}$$

Combining the assertion just proved with Lemma 3.2 (i) yields (6.1). \square

Definition 6.2. *For a vector $\vec{\gamma} > 0$ in \mathbb{R}^d , we denote by $\tilde{L}_{\vec{\gamma}}^r(\mathbb{R}^d)$ the set of all functions which are finite sums of functions of the form $m(\xi) \prod_{i=1}^d a_i(\xi_i)$, with $m \in L_{\vec{\gamma}}^r(\mathbb{R}^d)$ and $a_i(\xi_i)$ is either 1 or $\text{sgn}(\xi_i)$.*

We obtain below the boundedness of $\mathcal{E}_k^{(1)}$ by showing that $\sigma_k^{(n)}$ multiplied by a bump supported in an annulus around zero lies in $\tilde{L}_{\vec{\gamma}}^r(\mathbb{R}^{k+1})$ for appropriate r and $\vec{\gamma}$. Throughout this section, we have $\frac{2}{\gamma} < r < \frac{1}{\gamma-1}$, $1 < r \leq 2$, and $\vec{\gamma}/2 = (\gamma/2, \dots, \gamma/2)$ for all

variables, for some $1 < \gamma < 2$. Define

$$\begin{aligned} m_1(\xi_0, \xi_1) &= \operatorname{sgn}(\xi_1)(\Phi(\xi_0/\xi_1) - 1) \\ m_2(\xi_0, \xi_1) &= m(\xi_0, \xi_1) - m_1(\xi_0, \xi_1) = \operatorname{sgn}(\xi_1) \end{aligned}$$

so that $m_1(\xi_0, \xi_1) + m_2(\xi_0, \xi_1) = \operatorname{sgn}(\xi_1)\Phi(\xi_0/\xi_1)$. Recall that the multiplier $\sigma_k^{(1)}$ of $\mathcal{C}_k^{(1)}$ is

$$\sigma_k^{(1)}(\xi_0, \xi_1, \dots, \xi_k) = \prod_{l=1}^k \operatorname{sgn}(\xi_l)\Phi(\xi_0/\xi_l).$$

Having completed all this preliminary material, we now turn to the proof of Theorem 1.1 for all k, n .

Proof of Theorem 1.1. We first consider the case $n = 1$ and k is arbitrary. We decompose $\sigma_k^{(1)}$ as a finite sum of symbols σ_S , where each $\sigma_S(\xi_0, \dots, \xi_k)$ has the form $\prod_{l \in S} m_1(\xi_0, \xi_l) \prod_{l \notin S} \operatorname{sgn}(\xi_l)$ for some subset S of $\{1, \dots, k\}$ with $|S| = k_1 \leq k$, with the understanding that a product indexed by an empty set equals 1. We will prove that for all subsets S of $\{1, \dots, k\}$ with $|S| = k_1$ we have

$$(6.2) \quad \prod_{l \in S} m_1(\xi_0, \xi_l) \Psi_{k_1+1}(\vec{\xi}_S) \in \tilde{L}_{\vec{\gamma}/2}^r(\mathbb{R}^{k_1+1}), \quad \frac{2}{\gamma} < r < \min\left(\frac{1}{\gamma-1}, \frac{2k}{(k-1)\gamma}\right).$$

Here $\vec{\xi}_S$ is a vector formed by the variables of $(\xi_l)_{l \in S}$ and Ψ_{k_1+1} is a smooth bump supported in an annulus centered at zero in \mathbb{R}^{k_1+1} . Assuming (6.2), we prove Theorem 1.1 when $n = 1$. Indeed, to prove that $\mathcal{C}_k^{(1)}$ is bounded from $L^{p_0} \times \dots \times L^{p_k}$ to L^p , given $p_0, \dots, p_k \in (1, \infty)$, choose $\gamma < 2$ such that $\gamma > \frac{2k}{k+1}$ and $\gamma > \frac{2}{p_i}$ for all $i = 0, 1, \dots, k$. Then choose r such that $\frac{1}{\gamma-1} > r > \frac{2}{\gamma}$ and apply Corollary 2.2 to obtain Theorem 1.1 when $n = 1$.

We prove (6.2) by induction in k . The case $k = 1$ was proved in Proposition 5.6. Assume we have (6.2) for all $k' \leq k-1$ and we consider the case where $k' = k$.

If $\sigma_S = \prod_{l=1}^k \operatorname{sgn}(\xi_l)$, then the conclusion is trivial in view of the boundedness of the Hilbert transform. If σ_S contains at least one factor of m_2 but $\sigma_S \neq \prod_{l=1}^k \operatorname{sgn}(\xi_l)$, say for instance that $\sigma_S = \prod_{l=1}^{k_1} m_1(\xi_0, \xi_l) \prod_{l=k_1+1}^k \operatorname{sgn}(\xi_l)$, we observe that σ_S satisfies $\prod_{l=1}^{k_1} m_1(\xi_0, \xi_l) \Psi_{k_1+1} \in \tilde{L}_{\vec{\gamma}/2}^r(\mathbb{R}^{k_1+1})$ by the inductive hypothesis. Hence the corresponding operator T_{σ_S} is bounded as required, in view of Corollary 2.2.

Next we will discuss the most difficult case, namely $\sigma_S = \prod_{l=1}^k m_1(\xi_0, \xi_l)$.

We decompose $m_1 = B^0 + B^1 + G$ such that

$$\begin{aligned} B^0 &= m_1(\xi_0, \xi_1) \psi_0(\xi_0/\xi_1) = \operatorname{sgn}(\xi_1)(\Phi(\xi_0/\xi_1) - 1) \psi_0(\xi_0/\xi_1), \\ B^1 &= m_1(\xi_0, \xi_1) \psi_1(\xi_0/\xi_1) = \operatorname{sgn}(\xi_1)(\Phi(\xi_0/\xi_1) - 1) \psi_1(\xi_0/\xi_1) \end{aligned}$$

$$G = m_1(\xi_0, \xi_1)\psi_2(\xi_0/\xi_1) = \text{sgn}(\xi_1)(\Phi(\xi_0/\xi_1) - 1)\psi_2(\xi_0/\xi_1),$$

where ψ_0 and ψ_1 are smooth functions supported in a neighborhood of length $1/100$ centered at 0 and -1 respectively, and they take value 1 in a smaller neighborhood, while $\psi_2 = 1 - \psi_0 - \psi_1$. Then we write σ_S as a sum of 3^k terms where each term has k factors of the form

$$F_1(\xi_0, \xi_1)F_2(\xi_0, \xi_2) \cdots F_k(\xi_0, \xi_k)$$

where $F_1, F_2, F_3 \in \{B^0, B^1, G\}$. Let us denote one of these finitely many terms by σ_{ss} . Let Ψ_{k+1} be a smooth function supported in an annulus centered at zero in \mathbb{R}^{k+1} .

Lemma 6.3. $G(\xi_0, \xi_l)\theta_2(\xi_0, \xi_l)$ and $B^1(\xi_0, \xi_l)\theta_2(\xi_0, \xi_l)$ lie in $\tilde{L}_{\tilde{\gamma}/2}^r(\mathbb{R}^2)$.

Proof. Let $G_1 = \text{sgn}(\xi_1)\Phi(\xi_0/\xi_1)\psi_2(\xi_0/\xi_1)$ and $G_2 = \psi_2(\xi_0/\xi_1)$, then we have $G = G_1 - \text{sgn}(\xi_1)G_2$. It suffices to prove that $G_1\theta_2$ and $G_2\theta_2$ are Schwartz functions. Note that θ_2 is smooth and compactly supported, so it suffices to prove that G_1 and G_2 are smooth on the support of θ_2 . There are three possible places where G_1 or G_2 is not differentiable, namely $\xi_0/\xi_1 \sim -1$, $\xi_0/\xi_1 \sim 0$ and $\xi_1 \sim 0$. The first two cases do not occur because they fall outside the support of ψ_2 . For the last case, we notice that $G_1(\xi_0, \xi_1) = G_2(\xi_0, \xi_1) = 1$ on a neighborhood of the positive ξ_0 -axis, and $G_1(\xi_0, \xi_1) = -1 = -G_2(\xi_0, \xi_1)$ on a neighborhood of the negative ξ_0 neighborhood, where it is always smooth since ξ_0 is away from 0 .

Let $B_1^1(\xi_0, \xi_1) = \text{sgn}(\xi_1)\Phi(\xi_0/\xi_1)\psi_1(\xi_0/\xi_1)$ and $B_2^1(\xi_0, \xi_1) = \psi_1(\xi_0/\xi_1)$, then

$$B^1 = \text{sgn}(\xi_1)(\Phi(\xi_0/\xi_1) - 1)\psi_1(\xi_0/\xi_1) = B_1^1 - \text{sgn}(\xi_1)B_2^1.$$

But $B_1^1\theta_2 \in L_{\tilde{\gamma}}^r(\mathbb{R}^2)$ by the calculation in Section 5 and $B_2^1\theta_2$ is a Schwartz function. Hence $B^1\theta_2 \in \tilde{L}_{\tilde{\gamma}/2}^r(\mathbb{R}^2)$. \square

Lemma 6.4. If B^0 is not a factor in σ_{ss} , then $\sigma_{ss}\Psi_{k+1} \in \tilde{L}_{\tilde{\gamma}/2}^r(\mathbb{R}^{k+1})$.

Proof. If B^0 is not a factor in σ_{ss} , by the support of the terms involved, we easily get that $1 \lesssim_k |\xi_0|$, so we are allowed to insert bumps $\prod_{l=1}^k \theta_2(\xi_0, \xi_l)$, where θ_2 is smooth and is supported in annulus around the origin in \mathbb{R}^2 , without changing the value of $\sigma_{ss}\Psi_{k+1}$. Notice that $\sigma_l(\xi_0, \xi_l)\theta_2(\xi_0, \xi_l)$ lies in $\tilde{L}_{\tilde{\gamma}/2}^r(\mathbb{R}^2)$, where σ_l could be B^1 or G . Therefore, by Lemma 6.1 we have $\prod_{l=1}^k \sigma_l(\xi_0, \xi_l)\theta_2(\xi_0, \xi_l)$ lies in $\tilde{L}_{\tilde{\gamma}/2}^r(\mathbb{R}^{k+1})$ and, consequently, $\sigma_{ss}\Psi_{k+1}$ lies in $\tilde{L}_{\tilde{\gamma}/2}^r(\mathbb{R}^{k+1})$. \square

Before we consider the terms σ_{ss} containing at least one B^0 , we state a lemma.

Lemma 6.5. *Let*

$$A(\xi_0, \dots, \xi_{k-1}) = \prod_{l=1}^{k-1} \sigma_l(\xi_0, \xi_l) \Theta_k(\xi_0, \dots, \xi_{k-1}),$$

where σ_l is homogeneous of degree zero and Θ_k is supported in $|\xi_0| + \dots + |\xi_{k-1}| \leq 1$. Suppose that $A\Psi_k \in \tilde{L}_{\vec{\gamma}/2}^r(\mathbb{R}^k)$ for some Ψ_k smooth function supported in an annulus around the origin in \mathbb{R}^k and θ_1 is a smooth function supported in $[-2, -1/2] \cup [1/2, 2]$. Then we have $AB^0\theta_1(\xi_k)\Psi_{k+1} \in \tilde{L}_{\vec{\gamma}/2}^r(\mathbb{R}^{k+1})$, where Ψ_{k+1} is a smooth function supported in an annulus around the origin in \mathbb{R}^{k+1} .

Assuming momentarily the validity of the preceding lemma, we complete the proof of (6.2). It remains to consider the case where σ_{ss} contains a factor of B^0 . In this case, by symmetry, we may assume that the last factor of σ_{ss} is $B^0(\xi_0, \xi_k)$. Moreover, we may assume that in the support of $\sigma_{ss}\Psi_{k+1}$, we have $|\xi_k| \sim 1$. This is the case if there is only one factor of the form B^0 . If there exist two factors of the form B^0 in the product, say $B^0(\xi_0, \xi_{k-1})$ and $B^0(\xi_0, \xi_k)$, then we write

$$\sigma_{ss}\Psi_{k+1} = \sigma_{ss}\varphi_2(\xi_{k-1}, \xi_k)\Psi_{k+1} + \sigma_{ss}(1 - \varphi_2)(\xi_{k-1}, \xi_k)\Psi_{k+1},$$

where on the support of φ_2 we have $|\xi_k| \lesssim |\xi_{k-1}|$ and on the support of $1 - \varphi_2$ we have $|\xi_{k-1}| \lesssim |\xi_k|$. By symmetry, we consider only the term

$$\begin{aligned} & \sigma_{ss}(1 - \varphi_2)(\xi_{k-1}, \xi_k)\Psi_{k+1} \\ &= \left[\prod_{l=1}^{k-1} \sigma_l(\xi_0, \xi_l) \right] \Theta_k(\xi_0, \dots, \xi_{k-1}) B^0(\xi_0, \xi_k) \theta_1(\xi_k) (1 - \varphi_2)(\xi_{k-1}, \xi_k) \Psi_{k+1}, \end{aligned}$$

where we inserted a bump Θ_k which is supported in a ball centered at zero in \mathbb{R}^k and is equal to one on a smaller ball. By the inductive hypothesis we have that the function $\prod_{l=1}^{k-1} \sigma_l(\xi_0, \xi_l) \Theta_k(\xi_0, \dots, \xi_{k-1}) \Psi_k$ lies in $\tilde{L}_{\vec{\gamma}/2}^r(\mathbb{R}^k)$. Consequently, by Lemma 6.5, we have that $\sigma_{ss}(1 - \varphi_2)\Psi_{k+1}$ is in $\tilde{L}_{\vec{\gamma}/2}^r(\mathbb{R}^{k+1})$ for an appropriate choice of r, γ . Analogously $\sigma_{ss}\varphi_2\Psi_{k+1}$ is in $\tilde{L}_{\vec{\gamma}/2}^r(\mathbb{R}^{k+1})$ and thus so does $\sigma_{ss}\Psi_{k+1}$. \square

Proof of Lemma 6.5. By the definition of $\tilde{L}_{\vec{\gamma}}^r$, it suffices to consider the case $A\Psi_k$ lies in $L_{\vec{\gamma}/2}^r(\mathbb{R}^k)$. We write

$$AB^0\theta_1 = \sum_{j \leq 2} \Psi_k(2^{-j}\vec{\xi}') A(\vec{\xi}') B^0(\xi_0, \xi_k) \varphi(2^{-j}\xi_0) \theta_1(\xi_k)$$

where φ is supported in a ball around zero and $\vec{\xi}' = (\xi_0, \dots, \xi_{k-1})$. Applying the Kato-Ponce inequality, we have

$$\|AB^0\theta_1\|_{L_{\vec{\gamma}}^r(\mathbb{R}^{k+1})}$$

$$\begin{aligned}
& \leq \left\| \sum_{j \leq 2} \left\| J_0^{\frac{\gamma}{2}} \left[J_1^{\frac{\gamma}{2}} \cdots J_{k-1}^{\frac{\gamma}{2}} (\Psi_k(2^{-j} \cdot) A(\cdot)) J_k^{\frac{\gamma}{2}} (B^0(\varphi(2^{-j} \cdot) \otimes \theta_1)) \right] \right\|_{L^r(\xi_0)} \right\|_{L^r(\mathbb{R}^k)} \\
& \lesssim \sum_{j \leq 2} \left\| \left\| J_0^{\frac{\gamma}{2}} \left[J_1^{\frac{\gamma}{2}} \cdots J_{k-1}^{\frac{\gamma}{2}} (\Psi_k(2^{-j} \cdot) A(\cdot)) \right] \right\|_{L^r(\xi_0)} \left\| J_k^{\frac{\gamma}{2}} (B^0(\varphi(2^{-j} \cdot) \otimes \theta_1)) \right\|_{L^\infty(\xi_0)} \right\|_{L^r(\mathbb{R}^k)} \\
& \quad + \left\| J_1^{\frac{\gamma}{2}} \cdots J_{k-1}^{\frac{\gamma}{2}} (\Psi_k(2^{-j} \cdot) A(\cdot)) \right\|_{L^\infty(\xi_0)} \left\| J_0^{\frac{\gamma}{2}} \left[J_k^{\frac{\gamma}{2}} (B^0(\varphi(2^{-j} \cdot) \otimes \theta_1)) \right] \right\|_{L^r(\xi_0)} \right\|_{L^r(\mathbb{R}^k)} \\
(6.3) \quad & \leq C \sum_{j \leq 2} \|A \Psi_k(2^{-j} \cdot)\|_{L_{\frac{\gamma}{2}}^r(\mathbb{R}^k)} \|B^0 \varphi(2^{-j} \cdot) \otimes \theta_1\|_{L_{\frac{\gamma}{2}}^r(\mathbb{R}^2)},
\end{aligned}$$

where in the last inequality we used the Sobolev embedding theorem and the fact that $r\gamma > 2$. By a simple change of variables it follows that

$$\|A \Psi_k(2^{-j} \cdot)\|_{L_{\frac{\gamma}{2}}^r(\mathbb{R}^k)} = \left\| \left(\prod_{l=1}^{k-1} \sigma_l(\xi_0, \xi_l) \Psi_k \right) (2^{-j} \cdot) \right\|_{L_{\frac{\gamma}{2}}^r(\mathbb{R}^k)} \lesssim 2^{jk(\frac{1}{r} - \frac{\gamma}{2})}.$$

Let $\sigma_{b2j} = B^0(\varphi(2^{-j} \cdot) \otimes \theta_1)$. We need the following estimate about σ_{b2j}

$$(6.4) \quad \|\sigma_{b2j}\|_{L_{\frac{\gamma}{2}, \gamma/2}^r} \leq 2^{j\frac{\gamma}{2}}.$$

Assuming (6.4) we control (6.3) by

$$C \sum_{j \leq 2} 2^{jk(\frac{1}{r} - \frac{\gamma}{2})} 2^{j\frac{\gamma}{2}},$$

which is finite if we chose the pair r, γ such that

$$(6.5) \quad \frac{2}{\gamma} < r < \min\left(\frac{1}{\gamma-1}, \frac{2k}{(k-1)\gamma}\right).$$

We observe that $\frac{2k}{(k-1)\gamma}$ is decreasing in k , so any pair (r, γ) satisfying (6.5) for k also satisfies the same condition for any $k' \leq k$.

We now prove (6.4). It is easy to verify that

$$\begin{aligned}
\|\sigma_{b2j}\|_{L_{\frac{\gamma}{2}, \gamma/2}^r(\mathbb{R}^2)} & \lesssim \|\sigma_{b2j}\|_{L^r(\mathbb{R}^2)} + \|\partial_1^{\frac{\gamma}{2}} \sigma_{b2j}\|_{L^r(\mathbb{R}^2)} + \|\partial_2^{\frac{\gamma}{2}} \sigma_{b2j}\|_{L^r(\mathbb{R}^2)} + \|\partial_1^{\frac{\gamma}{2}} \partial_2^{\frac{\gamma}{2}} \sigma_{b2j}\|_{L^r(\mathbb{R}^2)} \\
& \lesssim \|\sigma_{b2j}\|_{L^r(\mathbb{R}^2)} + 2^{-j(\frac{\gamma}{2} - \frac{1}{r})} \|\sigma_{b2j}(2^j \xi_0, \xi_1)\|_{L_\gamma^r(\mathbb{R}^2)},
\end{aligned}$$

since $j \leq 2$, where we used the notation $\partial_1^{\frac{\gamma}{2}} g = (|\xi_1|^{\frac{\gamma}{2}} \widehat{g}(\xi_1, \xi_2))^\vee$. Moreover, we have $\|\sigma_{b2j}\|_{L^r} \leq C 2^{j+j/r}$ by a change of variables and from the fact that $|\Phi(2^j \xi_0 / \xi_1) - 1| \leq C 2^j$.

Note that in the support of $\sigma_{b2j}(2^j \xi_0, \xi_1)$ we have $|\xi_0| \leq 2$, so we can introduce a smooth bump Θ_1 which is compactly supported in $[-3, 3]$ and assumes the value 1 on $[-2, 2]$, then Lemma 5.2 gives that

$$\|\sigma_{b2j}(2^j \xi_0, \xi_1)\|_{L_\gamma^r(\mathbb{R}^2)} = \|\Theta_1(\xi_0 / \xi_1) (\Phi(2^j \xi_0 / \xi_1) - 1) \varphi(\xi_0) \theta_1(\xi_1)\|_{L_\gamma^r(\mathbb{R}^2)}$$

$$\begin{aligned} &\lesssim \left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1)\varphi(\xi_0 \xi_1)\theta_1(\xi_1) \right\|_{L_\gamma^r(\mathbb{R}^2)} \\ &\quad + \left\| \nabla[\Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1)\varphi(\xi_0 \xi_1)\theta_1(\xi_1)] \right\|_{L^\infty(\mathbb{R}^2)}. \end{aligned}$$

The term involving the L^∞ norm of the gradient is controlled by 2^j trivially. The Kato-Ponce inequality yields that

$$\begin{aligned} &\left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1)\varphi(\xi_0 \xi_1)\theta_1(\xi_1) \right\|_{L_\gamma^r(\mathbb{R}^2)} \\ &\lesssim \left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1)\theta_1(\xi_1) \right\|_{L_\gamma^r(\mathbb{R}^2)} \left\| \varphi(\xi_0 \xi_1)\theta_1(\xi_1) \right\|_{L^\infty(\mathbb{R}^2)} \\ &\quad + \left\| \varphi(\xi_0 \xi_1)\theta_1(\xi_1) \right\|_{L_\gamma^r(\mathbb{R}^2)} \left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1)\theta_1(\xi_1) \right\|_{L^\infty(\mathbb{R}^2)}. \end{aligned}$$

Trivially we have $\left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1)\theta_1(\xi_1) \right\|_{L^\infty(\mathbb{R}^2)} \leq C2^j$. By Lemma 5.3,

$$\left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1)\theta_1(\xi_1) \right\|_{L_\gamma^r(\mathbb{R}^2)} \lesssim \left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1) \right\|_{L_\gamma^r(\mathbb{R})}.$$

We now estimate the last term by controlling $\left\| \Theta_1(\Phi_j - 1) \right\|_{L_1^r}$ and $\left\| (\Theta_1(\Phi_j - 1))' \right\|_{L_\alpha^r}$ with $\alpha = \gamma - 1 \in (0, 1)$, where $\Phi_j(\xi_0) = \Phi(2^j \xi_0)$. Notice that $|\Phi(\xi_0) - 1| \leq C|\xi_0|$, so it is easy to verify that $\left\| \Theta_1(\xi_0)(\Phi(2^j \xi_0) - 1) \right\|_{L^r} \leq C2^j$. Also we have

$$(\Theta_1(\Phi_j - 1))' = \Theta_1'(\Phi_j - 1) + \Theta_1(\Phi_j)' =: g_1 + g_2.$$

Obviously $\|g_1\|_{L^r} + \|g_2\|_{L^r} \leq C2^j$, hence we have $\left\| \Theta_1(\Phi_j - 1) \right\|_{L_1^r} \leq C2^j$.

As for $\left\| (\Theta_1(\Phi_j - 1))' \right\|_{L_\alpha^r}$, let us consider $\|g_1'\|_{L_1^r}$ and $\|g_2\|_{L_\alpha^r}$. The former is controlled by $\left\| \Theta_1''(\Phi_j - 1) \right\|_{L^r} + \left\| \Theta_1'(\Phi_j)' \right\|_{L^r}$, which turns out to be less than $C2^j$. Applying the Kato-Ponce inequality ([7], [34], [19]) to $\|g_2\|_{L_\alpha^r}$ we obtain

$$\begin{aligned} \|g_2\|_{L_\alpha^r} &\lesssim \left\| \Theta_1 \right\|_{L^\infty} \left\| (\Phi_j)' \right\|_{L_\alpha^r} + \left\| \Theta_1 \right\|_{L_\alpha^r} \left\| (\Phi_j)' \right\|_{L^\infty} \\ &\lesssim 2^j 2^{j\alpha} 2^{-j/r} \left\| \partial^\alpha \Phi' \right\|_{L^r} + 2^j \\ &\lesssim 2^{j(\gamma - \frac{1}{r})} + 2^j. \end{aligned}$$

Then we have

$$\|g_1\|_{L_\alpha^r} \lesssim \|g_1\|_{L_1^r} \lesssim \|g_1\|_{L^r} + \|g_1\|_{L_1^r} \leq C2^j$$

and

$$\|g_2\|_{L_\alpha^r} \lesssim \|g_2\|_{L^r} + \|g_2\|_{L_\alpha^r} \leq C(2^{j(\gamma - \frac{1}{r})} + 2^j).$$

Recall that $2 < r < \frac{1}{\gamma-1}$, which implies that

$$0 < \frac{1}{r} < \gamma - \frac{1}{r} < 1.$$

Lemma 5.1 implies that $\left\| \Theta_1(\Phi_j - 1) \right\|_{L_{\gamma/2, \gamma/2}^r(\mathbb{R}^2)} \leq C2^{j(\gamma - \frac{1}{r})}$, which decreases as $j \rightarrow -\infty$. Therefore we have $\left\| \sigma_{b2j}(2^j \xi_0, \xi_1) \right\|_{L_{\gamma/2, \gamma/2}^r(\mathbb{R}^2)} \lesssim 2^{j\gamma - j/r}$ and $\left\| \sigma_{b2j} \right\|_{L_{\gamma/2, \gamma/2}^r(\mathbb{R}^2)} \lesssim 2^{j\gamma/2}$. Thus (6.4) holds, hence the proof of (6.2) is complete and so is the proof of Theorem 1.1 in the case $n = 1$.

We now prove Theorem 1.1 when $n \geq 2$ and k is arbitrary. In this case, for some finite sets S_k , we write

$$\begin{aligned}\sigma_k^{(n)} &= \prod_{l=1}^n \prod_{j=1}^k \operatorname{sgn}(\xi_{jl}) \Phi(\xi_{0l}/\xi_{jl}) \\ &= \prod_{l=1}^n \sigma_k^{(1)}(\xi_{0l}, \xi_{1l}, \dots, \xi_{kl}) \\ &= \prod_{l=1}^n \sum_{s \in S_k} \sigma_{k,s}(\xi_{0l}, \xi_{1l}, \dots, \xi_{kl}),\end{aligned}$$

where $\sigma_{k,s}$ is a product of two pieces, with the first piece being a function $\sigma'_{k,s}$ of $k_1 + 1$ variables with $k_1 \leq k$ such that $\sigma'_{k,s} \Psi \in \tilde{L}_\gamma^r(\mathbb{R}^{k_1+1})$, and the second piece being a product of $k - k_1$ signum functions. By the boundedness of the Hilbert transform, it is equivalent to consider the multiplier with the second piece replaced by 1. Since $\sigma_k^{(n)}$ is a finite sum of functions satisfying the hypotheses of Corollary 2.5, it follows that

$$\|\mathcal{C}_k^{(n)}(f_0, f_1, \dots, f_k)\|_{L^p(\mathbb{R}^n)} \leq C(n, k) \prod_{j=0}^k \|f_j\|_{L^{p_j}(\mathbb{R}^n)}.$$

□

7. FINAL REMARKS

In this work we have obtained boundedness for $\mathcal{C}_k^{(n)}$ in the largest possible open set of indices in $(\mathbb{R}^+)^{k+1}$. It remains to consider the endpoint cases where some p_j are equal to 1 or ∞ . We consider the situations where $n = 1$ and $n \geq 2$ separately:

Case: $n = 1$. In the case $k = 1$ we have the largest region on which boundedness holds for $\mathcal{C}_k^{(1)}$. This is an easy consequence of the results in [12], but was not explicitly stated in that work. We take the opportunity to state this result in the present work. As an application of Theorems 3.1 and 4.1 in [12] we obtain the following result:

Corollary 7.1. *For $1 \leq p_j \leq \infty$ for $j = 0, 1$, $1/p = 1/p_0 + 1/p_1$, and $1/2 \leq p < \infty$, the statements below are valid:*

(i) *when all $p_j > 1$, then $\mathcal{C}_1^{(1)}$ can be extended to be a bounded operator from the 2-fold product $L^{p_0}(\mathbb{R}) \times L^{p_1}(\mathbb{R})$ to $L^p(\mathbb{R})$;*

(ii) *when some $p_j = 1$, then $\mathcal{C}_1^{(1)}$ can be extended to be a bounded operator from the 2-fold product $L^{p_0}(\mathbb{R}) \times L^{p_1}(\mathbb{R})$ to $L^{p,\infty}(\mathbb{R})$. In particular we have*

$$\|\mathcal{C}_1^{(1)}\|_{L^1 \times L^1 \rightarrow L^{1/2,\infty}} < \infty.$$

In the case $k \geq 2$, we believe that boundedness for $\mathcal{C}_k^{(1)}$ holds in the endpoint cases where some p_j are equal to 1 or ∞ , and this question will be considered in the future.

Case: $n \geq 2$. We remark that in this case boundedness may not hold for every point of this boundary region. For instance, $\mathcal{C}_k^{(n)}$ does not map $L^1(\mathbb{R}^n) \times \cdots \times L^1(\mathbb{R}^n)$ to $L^{1/(k+1),\infty}(\mathbb{R}^n)$ when $n \geq 2$. To verify this assertion, for simplicity, we take $n = 2$ and $k = 1$ but the same idea works for all larger n and k . Let $f_0 = f_1 = \chi_{[0,1]^2}$. Then for $x_1, x_2 \geq 2$ we have

$$|\mathcal{C}_1^{(2)}(f_0, f_1)(x_1, x_2)| \approx x_1^{-2} x_2^{-2}$$

and we note that for $\alpha < 1/16$ we have

$$\left| \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 2, \quad x_1^{-2} x_2^{-2} > \alpha\} \right| = \frac{1}{\sqrt{\alpha}} \log \frac{1}{4\sqrt{\alpha}} - 2 \left(\frac{1}{2\sqrt{\alpha}} - 2 \right),$$

hence $\mathcal{C}_1^{(2)}(f_0, f_1)$ does not lie in $L^{1/2,\infty}(\mathbb{R}^2)$.

In the case $n \geq 2$, as mentioned in the introduction, boundedness for $\mathcal{C}_k^{(n)}$ holds in the endpoint cases where some indices are equal to infinity. General endpoint bounds for $\mathcal{C}_k^{(n)}$ extend beyond the scope of this work and will be considered in future studies.

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APPENDIX: AN ENDPOINT MULTILINEAR MULTIPLIER THEOREM

Let $H^1(\mathbb{R}^n)$ denote the classical Hardy space on \mathbb{R}^n . We give an endpoint case of Theorem 2.1 when $l = 1$, which is of interest in multilinear multiplier theory.

Theorem 7.2. *Let $\vec{\gamma} = (\bar{\gamma}_1, \dots, \bar{\gamma}_m)$, where $\bar{\gamma}_i = (\gamma_{i1}, \dots, \gamma_{in})$ for each $1 \leq i \leq m$ and suppose that $\gamma_{i\ell} > 1$ for all $i = 1, \dots, m$ and $\ell = 1, \dots, n$. Let $1 < r \leq 2$ for all $i = 1, \dots, m$ and let σ be a bounded function on \mathbb{R}^{mn} such that*

$$\sup_{j \in \mathbb{Z}} \left\| \sigma(2^j \cdot) \widehat{\Psi} \right\|_{L^r_{\vec{\gamma}}(\mathbb{R}^{mn})} = A < \infty,$$

where $\widehat{\Psi}$ is a smooth function supported in the annulus $\frac{1}{2} \leq |\vec{\xi}| \leq 2$ in \mathbb{R}^{mn} satisfying $\sum_{j \in \mathbb{Z}} \widehat{\Psi}(2^{-j} \vec{\xi}) = 1$ for all $\vec{\xi} \in \mathbb{R}^{mn} \setminus \{0\}$. Then we have

$$(7.6) \quad \|T_\sigma\|_{H^1(\mathbb{R}^n) \times \dots \times H^1(\mathbb{R}^n) \rightarrow L^{1/m, \infty}(\mathbb{R}^n)} \lesssim A.$$

We note that when $m = 1$, boundedness for T_σ is known to hold from H^1 to L^1 .

Proof. For $1 \leq k \neq l \leq m$, recall the sets $U_{k,l}$ and $W_{k,l}$ and the functions $\phi_{k,l}$ and $\psi_{k,l}$ in the proof of Theorem 2.1. Letting $\sigma_{k,l}^1 = \sigma \phi_{k,l}$ and $\sigma_{k,l}^2 = \sigma \psi_{k,l}$, we write

$$\sigma = \sum_{1 \leq k \neq l \leq m} (\sigma_{k,l}^1 + \sigma_{k,l}^2).$$

By the symmetry, it suffices to consider the case where $k = m - 1$ and $l = m$. We establish the claimed estimate for T_{σ_1} and T_{σ_2} with $\sigma_1 = \sigma_{m-1,m}^1$ and $\sigma_2 = \sigma_{m-1,m}^2$.

We first consider $T_{\sigma_1}(f_1, \dots, f_m)$, where f_j are fixed Schwartz functions. We will prove

$$(7.7) \quad \|T_{\sigma_1}(f_1, \dots, f_m)\|_{H^{1/m, \infty}(\mathbb{R}^n)} \lesssim A \|f_1\|_{H^1(\mathbb{R}^n)} \cdots \|f_m\|_{H^1(\mathbb{R}^n)}.$$

Here $H^{1/m, \infty}$ denotes the weak Hardy space of all bounded tempered distributions whose smooth maximal function lies in the weak $L^{1/m}$. For f_j Schwartz functions, we have the following auxiliary estimate $\|T_{\sigma_1}(f_1, \dots, f_m)\|_{L^2(\mathbb{R}^n)} < \infty$ given in the proof of Theorem 2.1. Given $0 < p < \infty$, for F in $L^2(\mathbb{R}^n)$ there is a polynomial Q on \mathbb{R}^n such that

$$(7.8) \quad \|F - Q\|_{L^{p, \infty}(\mathbb{R}^n)} \leq C_{p,n} \|F - Q\|_{H^{p, \infty}(\mathbb{R}^n)} \approx \left\| \left(\sum_j |\Delta_j(F)|^2 \right)^{1/2} \right\|_{L^{p, \infty}(\mathbb{R}^n)},$$

by a result He [23]. But the fact that F lies in L^2 yields that $Q = 0$ (see the discussion in [23] on this). Hence estimate and (7.7) implies (7.6) via (7.8) with $Q = 0$.

Fix a Schwartz function θ whose Fourier transform is supported in the annulus $\frac{1}{2} \leq |\xi| \leq 2$ and $\sum_{j \in \mathbb{Z}} \widehat{\theta}(2^{-j} \xi) = 1$ for $\xi \in \mathbb{R}^n \setminus \{0\}$. Associated with θ we define the

Littlewood–Paley operator $\Delta_j^\theta(f) = f * \theta_{2^{-j}}$, where $\theta_t(x) = t^{-n}\theta(t^{-1}x)$ for $t > 0$. The function θ can be extended to the function Θ defined on \mathbb{R}^{nm} by setting $\widehat{\Theta}(\vec{\xi}) = \widehat{\Theta}(\xi_1, \dots, \xi_m) = \widehat{\theta}(\xi_1 + \dots + \xi_m)$. Now we have

$$\begin{aligned} \Delta_j^\theta(T_{\sigma_1}(f_1, \dots, f_m))(x) &= \int_{\mathbb{R}^{mn}} \widehat{\theta}(2^{-j}(\xi_1 + \dots + \xi_m)) \sigma_1(\vec{\xi}) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_m)} d\vec{\xi} \\ &= \int_{\mathbb{R}^{mn}} \widehat{\Theta}(2^{-j}\vec{\xi}) \sigma_1(\vec{\xi}) \widehat{f_1}(\xi_1) \cdots \widehat{f_m}(\xi_m) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_m)} d\vec{\xi}. \end{aligned}$$

Note that for all $\vec{\xi} = (\xi_1, \dots, \xi_m)$ in the support of the function $\widehat{\Theta}(2^{-j}\vec{\xi}) \sigma_1(\vec{\xi})$, we always have $2^{j-2} \leq |\xi_m| \leq 2^{j+2}$. Therefore we can take a Schwartz function η whose Fourier transform is supported in $\frac{1}{8} \leq |\xi| \leq 8$ and identical to 1 on $\frac{1}{4} \leq |\xi| \leq 4$ and insert the factor $\widehat{\eta}(2^{-j}\xi_m)$ into the above integral without changing the outcome. More specifically

$$\begin{aligned} \Delta_j^\theta(T_{\sigma_1}(f_1, \dots, f_m))(x) &= \int_{\mathbb{R}^{mn}} \widehat{\Theta}(2^{-j}\vec{\xi}) \sigma_1(\vec{\xi}) \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{\eta}(2^{-j}\xi_m) \widehat{f_m}(\xi_m) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_m)} d\vec{\xi}. \end{aligned}$$

Now define $\widehat{\Psi}_*(\vec{\xi}) = \sum_{|k| \leq 4} \widehat{\Psi}(2^{-k}\vec{\xi})$ and note that $\widehat{\Psi}_*(2^{-j}\vec{\xi})$ is equal to 1 on the annulus $\{\vec{\xi} \in \mathbb{R}^{mn} : 2^{j-4} \leq |\vec{\xi}| \leq 2^{j+4}\}$ which contains the support of $\sigma_1 \widehat{\Theta}(2^{-j}\cdot)$. Then we write

$$\begin{aligned} \Delta_j^\theta(T_{\sigma_1}(f_1, \dots, f_m))(x) &= \int_{\mathbb{R}^{mn}} \widehat{\Psi}_*(2^{-j}\vec{\xi}) \widehat{\Theta}(2^{-j}\vec{\xi}) \sigma_1(\vec{\xi}) \widehat{f_1}(\xi_1) \cdots \widehat{f_{m-1}}(\xi_{m-1}) \widehat{\eta}(2^{-j}\xi_m) \widehat{f_m}(\xi_m) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_m)} d\vec{\xi}. \end{aligned}$$

Taking the inverse Fourier transform, we obtain that $\Delta_j^\theta(T_{\sigma_1}(f_1, \dots, f_m))(x)$ is equal to

$$\int_{(\mathbb{R}^n)^m} 2^{mnj} (\sigma_1^j \widehat{\Psi}_* \widehat{\Theta})^\vee(2^j(x - y_1), \dots, 2^j(x - y_m)) \prod_{i=1}^{m-1} f_i(y_i) \Delta_j^\eta(f_m)(y_m) d\vec{y},$$

where $d\vec{y} = dy_1 \cdots dy_m$, and $\sigma_1^j(\xi_1, \xi_2, \dots, \xi_m) = \sigma_1(2^j \xi_1, 2^j \xi_2, \dots, 2^j \xi_m)$.

By Lemma 4.1 we now have

$$\begin{aligned} |\Delta_j^\theta(T_{\sigma_1}(f_1, \dots, f_m))(x)| &\leq \int_{(\mathbb{R}^n)^m} \prod_{i=1}^m \omega_{\bar{\gamma}_i}(2^j(x - y_i)) |(\sigma_1^j \widehat{\Psi}_* \widehat{\Theta})^\vee(2^j(x - y_1), \dots, 2^j(x - y_m))| \\ &\quad \times \frac{2^{mnj} |f_1(y_1)| \cdots |f_{m-1}(y_{m-1})| |\Delta_j^\eta(f_m)(y_m)|}{\prod_{i=1}^m \omega_{\bar{\gamma}_i}(2^j(x - y_i))} d\vec{y} \\ (7.9) \quad &\lesssim \left\| \left(\prod_{i=1}^m \omega_{\bar{\gamma}_i} \right) (\sigma_1^j \widehat{\Psi}_* \widehat{\Theta})^\vee \right\|_{L^\infty} \left(\prod_{i=1}^{m-1} M(f_i)(x) \right) M(\Delta_j^\eta(f_m))(x), \end{aligned}$$

as a consequence of the fact that $\bar{\gamma}_i > 1$ for all $1 \leq i \leq m$.

For the first term in (7.9) we have

$$\left\| \left(\prod_{i=1}^m \omega_{\gamma_i} \right) (\sigma_1^j \widehat{\Psi}_* \widehat{\Theta})^\vee \right\|_{L^\infty} \leq \|\sigma_1^j \widehat{\Psi}_* \widehat{\Theta}\|_{L_{\vec{\gamma}}^r} \leq \sigma_1^j \|\widehat{\Psi}_* \widehat{\Theta}\|_{L_{\vec{\gamma}}^r} \lesssim \sup_{j \in \mathbb{Z}} \|\sigma^j \widehat{\Psi}_*\|_{L_{\vec{\gamma}}^r} \lesssim A$$

in view of (3.3), (3.4), the homogeneity of $\phi_{m-1,m}$, and Lemma 3.2 (ii). Thus, we proved

$$|\Delta_j^\theta(T_{\sigma_1}(f_1, \dots, f_m))| \lesssim A \left(\prod_{i=1}^{m-1} M(f_i) \right) M(\Delta_j^\eta(f_m)).$$

We prove estimate (7.6) for σ_1 . Using the preceding inequality we obtain

$$\begin{aligned} & \|T_{\sigma_1}(f_1, \dots, f_{m-1}, f_m)\|_{H^{1/m, \infty}(\mathbb{R}^n)} \\ & \lesssim \left\| \left\{ \sum_j |\Delta_j^\theta(T_{\sigma_1}(f_1, \dots, f_m))|^2 \right\}^{\frac{1}{2}} \right\|_{L^{1/m, \infty}(\mathbb{R}^n)} \\ & \lesssim A \left\| \left\{ \sum_j M(\Delta_j^\eta(f_m))^2 \right\}^{\frac{1}{2}} \right\|_{L^{1, \infty}(\mathbb{R}^n)} \prod_{i=1}^{m-1} \|M(f_i)\|_{L^{1, \infty}(\mathbb{R}^n)} \\ & \lesssim A \left\| \left\{ \sum_j |\Delta_j^\eta(f_m)|^2 \right\}^{\frac{1}{2}} \right\|_{L^1(\mathbb{R}^n)} \prod_{i=1}^{m-1} \|f_i\|_{L^1(\mathbb{R}^n)} \\ & \lesssim A \prod_{i=1}^m \|f_i\|_{H^1(\mathbb{R}^n)}. \end{aligned}$$

Next we deal with σ_2 . Using the notation introduced earlier, we write

$$T_{\sigma_2}(f_1, \dots, f_{m-1}, f_m) = \sum_{j \in \mathbb{Z}} T_{\sigma_2}(f_1, \dots, f_{m-1}, \Delta_j^\theta(f_m)).$$

We introduce another Littlewood–Paley operator Δ_j^ζ , which is given on the Fourier transform by multiplying with a bump $\widehat{\zeta}(2^{-j}\xi)$, where $\widehat{\zeta}$ is equal to one on the annulus $\{\xi \in \mathbb{R}^n : \frac{1}{2^k} \leq |\xi| \leq 4\}$ with $\frac{1}{2^k} \leq \frac{1}{20m}$, vanishes off the annulus $\frac{1}{2^{k+1}} \leq |\xi| \leq 8$, and $\sum_j \widehat{\zeta}(2^{-j}\xi) = k + 3$. The key observation in this case is that for each $j \in \mathbb{Z}$ we have

$$T_{\sigma_2}(f_1, \dots, f_{m-1}, \Delta_j^\theta(f_m)) = T_{\sigma_2}(f_1, \dots, f_{m-2}, \Delta_j^\zeta(f_{m-1}), \Delta_j^\theta(f_m)).$$

As in the previous case, the function under the following integral

$$\begin{aligned} & T_{\sigma_2}(f_1, \dots, f_{m-2}, \Delta_j^\zeta(f_{m-1}), \Delta_j^\theta(f_m))(x) \\ & = \int_{(\mathbb{R}^n)^m} \sigma_2(\vec{\xi}) \prod_{l=1}^{m-2} \widehat{f_l}(\xi_l) \widehat{\Delta_j^\zeta(f_{m-1})}(\xi_{m-1}) \widehat{\Delta_j^\theta(f_m)}(\xi_m) e^{2\pi i x \cdot (\xi_1 + \dots + \xi_m)} d\vec{\xi} \end{aligned}$$

is supported in $\frac{1}{2} \cdot 2^j \leq |\xi_1| + \dots + |\xi_m| \leq \frac{11m}{5} \cdot 2^j$. Thus one may insert the factor $\widehat{\Psi}(2^{-j-k}\xi_1, \dots, 2^{-j-k}\xi_m)$ into the integrand.

A similar calculation as in the case for σ_1 yields the estimate

$$|T_{\sigma_2}(f_1, \dots, f_{m-2}, \Delta_j^\zeta(f_{m-1}), \Delta_j^\theta(f_m))| \lesssim A \left(\prod_{i=1}^{m-2} M(f_i) \right) M(\Delta_j^\zeta(f_{m-1})) M(\Delta_j^\theta(f_m)).$$

Summing over j , taking $L^{1/m, \infty}$ quasinorms and applying the Littlewood-Paley characterization of H^1 we deduce

$$\begin{aligned} & \|T_{\sigma_2}(f_1, \dots, f_{m-1}, f_m)\|_{L^{1/m, \infty}(\mathbb{R}^n)} \\ & \lesssim A \left\| \prod_{i=1}^{m-2} M(f_i) \sum_{j \in \mathbb{Z}} M(\Delta_j^\zeta(f_{m-1})) M(\Delta_j^\theta(f_m)) \right\|_{L^{1/m, \infty}(\mathbb{R}^n)} \\ & \lesssim A \left\| \left\{ \prod_{i=1}^{m-2} M(f_i) \right\} \left\{ \sum_{j \in \mathbb{Z}} M(\Delta_j^\zeta(f_{m-1}))^2 \right\}^{\frac{1}{2}} \left\{ \sum_{j \in \mathbb{Z}} M(\Delta_j^\theta(f_m))^2 \right\}^{\frac{1}{2}} \right\|_{L^{1/m, \infty}(\mathbb{R}^n)} \\ & \lesssim A \left(\prod_{i=1}^{m-2} \|M(f_i)\|_{L^{1, \infty}} \right) \left\| \left\{ \sum_{j \in \mathbb{Z}} M(\Delta_j^\zeta(f_{m-1}))^2 \right\}^{\frac{1}{2}} \right\|_{L^{1, \infty}} \left\| \left\{ \sum_{j \in \mathbb{Z}} M(\Delta_j^\theta(f_m))^2 \right\}^{\frac{1}{2}} \right\|_{L^{1, \infty}} \\ & \lesssim A \left(\prod_{i=1}^{m-2} \|f_i\|_{L^1(\mathbb{R}^n)} \right) \left\| \left\{ \sum_{j \in \mathbb{Z}} |\Delta_j^\zeta(f_{m-1})|^2 \right\}^{\frac{1}{2}} \right\|_{L^1(\mathbb{R}^n)} \left\| \left\{ \sum_{j \in \mathbb{Z}} |\Delta_j^\theta(f_m)|^2 \right\}^{\frac{1}{2}} \right\|_{L^1(\mathbb{R}^n)} \\ & \lesssim A \prod_{i=1}^m \|f_i\|_{H^1(\mathbb{R}^n)}. \end{aligned}$$

This concludes the proof of Theorem 7.2. □

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